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# SOLUTIONS

OF

# WEEKLY PROBLEM PAPERS

BY THE

REV. JOHN J. MILNE, M.A.

London:

MACMILLAN AND CO.

1885.

P.

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RICHARD CLAY & SONS,
BREAD STREET HILL, LONDON, E.C.
*and Bungay, Suffolk.*

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# PREFACE.

My purpose in bringing out the present volume is to fulfil the promise made in the Preface of the Weekly Problem Papers, and to place before students the various methods which are serviceable in solving problems in Elementary Mathematics, many of which methods are not to be found in the ordinary text-books. My object being therefore both to increase a student's stock of mathematical knowledge, and to teach him to put it to a practical use, I have in many cases given two different solutions of a problem when I thought it would be to the advantage of the reader to do so. In almost every case I have given a preference to the methods of Elementary Geometry, as I think there is a tendency at the present time to allow them to be to some extent supplanted by those of Modern Geometry, which, although more fascinating, are scarcely as valuable a training to a student previous to his entering the regions of higher mathematics.

With regard to the arrangement adopted in the following pages, a little explanation is necessary. Some years ago I began to form Problem Papers for the use of my pupils, without any

view to publication, and I selected the most suitable questions which I met with from time to time. Amongst other sources I drew largely from the Tripos Papers of 1875 and 1878, which are generally acknowledged to contain some of the most interesting of the problems which have appeared of late years, and solutions of these questions have been brought out by the examiners. When I decided to print my collection of Problem Papers, with solutions, two courses were open to me, viz. either to insert the questions selected from these two Triposes, and merely to indicate their origin, or to omit them altogether. Owing to their instructive character I was unwilling to adopt the latter course, and I felt that I should be doing good service both to teachers and to students in bringing these questions before their notice. For the use of those who do not possess the solutions of these two Triposes, I have added in an Appendix an equal number of alternative questions which are to a great extent similar in character to the corresponding problems. I have also added two notes, one on Geometrical Maxima and Minima, and the other on the geometrical method of investigating the envelope of a line which moves subject to certain conditions. The problems bearing upon these two subjects are extremely interesting, although hints as to their treatment are seldom to be met with.

My thanks are due to my former pupils and to my other friends who have rendered me great assistance, but especially

to Mr. R. F. DAVIS, M.A., late of Queens' College, Cambridge, who supplied me with many very neat solutions, particularly in geometrical problems. Those marked with an asterisk are all due to him, and it is a source of great regret to me that I was not able to avail myself still further of his aid.

Owing to the nature of the present work it is impossible for me to expect that it is quite free from errors due to transcription. It is hoped, however, that the context will allow them to be corrected without difficulty. At the same time I should be glad to receive notice of any errors, or of suggested improvements in any of the methods employed.

On the following page will be found a list of the errors which occur in the WEEKLY PROBLEM PAPERS, most of which were due to the papers from which they were taken, and were detected on working out the solutions.

JOHN J. MILNE.

CHESTNUT HOUSE,
HEVERSHAM, MILNTHORPE.

# ERRATA IN WEEKLY PROBLEM PAPERS.

P. 4, last line, for $a_0 + a_2$ read $a_0 + a_1$.

P. 9, last line but two, for $a_3$ read $a_3$.

P. 15, last line, for $a = \beta$ read $a - \beta$.

P. 16, 2nd line, for $\left(-\frac{b}{2c}\right)$ read $\left(-\frac{b}{2c}\right)^n$.

| | | |
|---|---|---|
| PAPER II. | No. 6, | instead of the word 'in,' substitute 'consecutive terms of a fixed.' |
| „ VIII. | „ 7, | in the last factor, for $+ a^2$ read $- a^2$. |
| „ XII. | „ 2, | in first line, for $x$ read $x^n$. |
| „ „ | „ 7, | in last line, for $a \sin(a - a')$ read $a \sin(a - a')$. |
| „ XIV. | „ 4, | for $\cos 7\theta$ read $\cos 5\theta$. |
| „ XVI. | „ 3, | $\log 3 = \cdot 47712$. See note at end of solution. |
| „ XVII. | „ 2, | in the last line but one, for $x = -$, read $x = +$. |
| „ XXV. | „ 5, | last line, for 'intersect' read 'intersects.' |
| „ XXVII. | „ 6, | before the word 'Shew' insert 'The paper is on the point of falling over.' |
| „ XXIX. | „ 2, | for $\frac{1}{x} - c^2$ read $\frac{1}{z} - c^2$. |
| „ „ | „ 7, | the last line should be $\frac{1}{(a\alpha)^2} + \frac{1}{(b\beta)^2} = \frac{1}{c^4}$. |
| „ XXXIX. | „ 2, | the first line should be<br>If $p, q, r$ are all unequal positive integers, and $x$ is positive and not equal to unity. |
| „ XLI. | „ 6, | for 'axes' read 'semi-axes.' |
| „ XLII. | „ 1, | for $2nx(1 + x)^{n-1}$ read $2nx(1 + x)^{2n-1}$. |

| | | | |
|---|---|---|---|
| PAPER | XLIV. | No. | 6, after 'of' insert 'the intersection of.' |
| „ | XLVII. | „ | 1, last line, for $\frac{2}{4}$ read $\frac{4}{3}$. |
| „ | „ | „ | 6, third line, for $2a$ read $2a$. |
| „ | L. | „ | 1, in last line, for $l = m = n = \pm\frac{1}{3}$ read 'The real values of $l$, $m$, $n$ are given by $-2l = m = n = \pm\frac{2}{3}$. |
| „ | LII. | „ | 1, for 576 read 2304. |
| „ | LV. | „ | 1, fourth line, for 'index' read 'radix.' |
| „ | LVII. | „ | 1. (3), for 63 read 61. |
| „ | „ | „ | 2, for $\frac{P_n^2}{Q_{2n}}$ read $\frac{P_{2n}}{Q_{2n}}$. |
| „ | LVIII. | „ | 3, for $n\pi$ read $2n\pi$. |
| „ | „ | „ | 6, at the end add 'which also touches the chord.' |
| „ | LXI. | „ | 6, first line, for 'conjugate' read 'any two.' |
| „ | LXXIII. | „ | 5, at the end add 'and if $KM$ and $QN$ intersect in $R'$, shew that $QR' = QN$.' |
| „ | LXXIV. | „ | 1, for 7 read 5. |
| „ | LXXXVII. | „ | 1, first line, for $b$ read $a + b$. |
| „ | „ | „ | 2, insert 2 in the L of the 2nd fraction. |
| „ | XCV. | „ | 1. (3), this series should be $\frac{1}{1^2.3^2} + \frac{2}{3^2.5^2} + \frac{3}{5^2.7^2} + \frac{4}{7^2.9^2} + \ldots$ |
| „ | XCIX. | „ | 2, second line, for $\sec\frac{C}{2}$ read $\sin\frac{C}{2}$. |
| „ | „ | „ | 5, last line, for 'an ellipse' read 'a conic.' |

# ERRATA IN SOLUTIONS.

PAPER II. No. 2. (1), the denominator of the second fraction in each line should be $cx + a$.

Page 9, last line but one, all the + signs should be −.

„ „ last line, for + write $+ (-1)^n$.

„ 11, last line but three, for 309 read 306, and make the same correction in the last line.

PAPER VIII. No. 3, second line, for $AB$ read $AC$.

„ X. „ 2, first line, the numbers should be $1^3, 2^3 \ldots$

„ XIII. „ 2. (1), last line but one, for $b + y$ read $b + y(3b + 1)$.

„ XIV. „ 4, first line, for $\sin 2\theta - \sin\theta$ read $\sin 2\theta - \sin 0$.

„ „ „ „ last line but one, for $\dfrac{\sin 2^n\theta}{\sin\theta}$ read $\dfrac{\sin 2^n\theta}{2\sin\theta}$.

Page 38, last line but three, for $BD$ read $BC$.

PAPER XVIII. No. 4, for $s - a$ read $2(s - a)$.

„ XXI. „ 7, first line, dele $QC$.

„ XXV. „ 5, in the 6th line, $C$ is the intersection of $AB$ and $RS$.

„ XXVI. „ 2, after $4xyzw$ insert $+ xyz(x + y + z)$.

„ XXVII. „ 2, fourteenth line, for $a^2\gamma'^2$ read $a^2\gamma'^2$.

„ „ „ „ in the next line read $a' = \gamma = a'd - ad'$.

PAPER LXXIV. No. 1, last line, for 7 read 5.

Page 286, LXXIX. 3. (2), this should be

$$2 = \frac{1}{\cos^2\frac{\theta}{2}} + \frac{1}{2}\frac{\cos\theta}{\cos^2\frac{\theta}{2}\cos^2\frac{\theta}{2^2}} + \frac{1}{2^2}\frac{\cos\theta\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}\cos^2\frac{\theta}{2^2}\cos^2\frac{\theta}{2^3}} + \ldots$$

Page 292, XVIII. 1, seventh line, for 'quicker' read 'longer.'

„ 294, „ 7, at the end insert 'For geometrical proof see Besant, *Rect. Hyp.* Props. 9, 10.'

„ „ XIX. 2. (1), for 3 read 6 throughout.

# SOLUTIONS

OF

# WEEKLY PROBLEM PAPERS.

## Paper I.

1. The value of the bill is £506 5*s.*

∴ value of assets is £1423 0*s.* 4*d.*

∴ £2,134 10*s.* 6*d.* : £1,423 0*s.* 4*d.* :: 20*s.* : 13*s.* 4*d.* *Ans.*

2. $as+bc=(a+b)(a+c)$, $bs+ca=(b+c)(b+a)$, $cs+ab=(c+a)(c+b)$,
∴ $(as+bc)(bs+ca)(cs+ab)=(a+b)^2(b+c)^2(c+a)^2$.

3. $2^{\frac{1}{3}}x+2^{\frac{2}{3}}+1 \mid 2x^3-6x+5 \mid 2^{\frac{2}{3}}x^2-2^{\frac{1}{3}}(2^{\frac{2}{3}}+1)x+2^{\frac{4}{3}}-2^{\frac{2}{3}}+1$

$$\frac{2x^3+(2^{\frac{4}{3}}+2^{\frac{2}{3}})x^2}{-2^{\frac{2}{3}}(2^{\frac{1}{3}}+1)x^2-6x}$$

$$\frac{-2^{\frac{2}{3}}(2^{\frac{2}{3}}+1)x^2-2(2^{\frac{2}{3}}+1)x-2^{\frac{1}{3}}(2^{\frac{2}{3}}+1)x}{2^{\frac{1}{3}}(2^{\frac{1}{3}}-2^{\frac{2}{3}}+1)x+5}$$

$$2^{\frac{1}{3}}(2^{\frac{1}{3}}-2^{\frac{2}{3}}+1)x+2^2-2^{\frac{1}{3}}+2^{\frac{2}{3}}+2^{\frac{1}{3}}-2^{\frac{2}{3}}+1.$$

4. $1+2\cos(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta)$
$=1+\cos(\beta-\gamma)\{\cos(\gamma-\beta)+\cos(\gamma+\beta-2\alpha)\}$
$=\cos^2(\beta-\gamma)+1+\frac{1}{2}\{\cos 2(\beta-\alpha)+\cos 2(\alpha-\gamma)\}$
$=\cos^2(\beta-\gamma)+1+\cos^2(\beta-\alpha)-\frac{1}{2}+\cos^2(\alpha-\gamma)-\frac{1}{2}$
$=\cos^2(\beta-\gamma)+\cos^2(\gamma-\alpha)+\cos^2(\alpha-\beta)$.

5. Let the bisector of $A$ meet $BC$ in $D$.

Then $\frac{BD}{p} = \frac{\sin\frac{A}{2}}{\sin B}$, $\frac{DC}{p} = \frac{\sin\frac{A}{2}}{\sin C}$, $\therefore \frac{a}{p} = \sin\frac{A}{2}\left\{\frac{1}{\sin B} + \frac{1}{\sin C}\right\}$,

$\therefore \frac{\cos\frac{A}{2}}{p} = \frac{1}{2a}\left(\frac{\sin A}{\sin B} + \frac{\sin A}{\sin C}\right) = \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right)$

$\frac{\cos\frac{B}{2}}{q} = \frac{1}{2}\left(\frac{1}{c} + \frac{1}{a}\right)$, $\frac{\cos\frac{C}{2}}{r} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$,

$\therefore$ by addition the required result follows.

6. Let $DC$ cut the circles in $G$ and $H$.

Then $DE^2 = DG \,.\, DC$, $DF^2 = DC \,.\, DH$ . . . . . (1)

Since $\angle DCB = ACG = AGC$, $\therefore$ the triangles $DCB$, $DAG$ are similar

$\therefore BC \,.\, DG = AG \,.\, DC = AC \,.\, DC$ . . . . . (2)

So $AC \,.\, DH = BH \,.\, DC = BC \,.\, DC$ . . . . . (3)

$\therefore$ by (2), (3) $DG \,.\, DH = DC^2$; and by (1) $DE \,.\, DF = DC^2$.

7. Let the tangent at $P$ meet the major axis in $T'$.

Then $CA^2 : CS^2 :: CN : CG :: CN \,.\, CT' : CG \,.\, CT'$ $\therefore CG \,.\, CT' = CS^2$,

$\therefore \frac{CS}{CG} - 1 = \frac{CT'}{CS} - 1$, $\therefore \frac{ST'}{CS} = \frac{SG}{CG} = \frac{SP}{CR}$ $\therefore \frac{ST'}{SP} = \frac{CH}{CR}$

$\therefore RH$ is parallel to $PT'$.

Now $CT$ is parallel to $SP$, $\therefore CT = CA$; $\therefore HT$ is perpendicular to $PT'$, and $\therefore$ parallel to $PG$. $\therefore RHTP$ is a rectangle, $\therefore$ its diagonals are equal, $\therefore RT = HP$.

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## Paper II.

1. $x^n - nx + n - 1 \mid nx^{n+1} - (n+1)x^n + 1 \mid nx - (n+1)$

$nx^{n+1} - n^2x^2 + n^2x - nx$

$-(n+1)x^n + n^2x^2 - n^2x + nx + 1$

$-(n+1)x^n + n(n+1)x - n(n+1) + n + 1.$

$n^2 \mid n^2x^2 - 2n^2x + n^2$

$x^2 - 2x + 1$

Now $x^n - nx + n - 1 = x^n - 1 - n(x - 1)$

$$= (x - 1)\{x^{n-1} + x^{n-2} + \ldots + 1 - n\}$$

and if we put $x = 1$, the exp. in the 2nd bracket $= 0$

$\therefore (x - 1)^2$ is a factor of $x^n - nx + n - 1$

$\therefore (x - 1)^2$ is the G.C.M.

2. (1) $\dfrac{ax + b}{cx + b} + \dfrac{bx + a}{cx + a} = \dfrac{(a + b)(x + 2)}{cx + a + b}$

$$\therefore \frac{cx + b + x(a - c)}{cx + b} + \frac{cx + a + x(b - c)}{cx + b} = \frac{2(cx + a + b) + x(a + b - 2c)}{cx + a + b}$$

$$\therefore 1 + \frac{x(a - c)}{cx + b} + 1 + \frac{x(b - c)}{cx + b} = 2 + \frac{x(a + b - 2c)}{cx + a + b}$$

$$\therefore \frac{x(a - c)}{cx + b} + \frac{x(b - c)}{cx + b} = \frac{x(a + b - 2c)}{cx + a + b}$$

$\therefore$ either $x = 0$

or $\dfrac{a - c}{cx + b} + \dfrac{b - c}{cx + b} = \dfrac{a + b - 2c}{cx + a + b}$

a simple equation, which gives $x = -\dfrac{a^3 + b^3 - c(a^2 + b^2)}{c\{a^2 + b^2 - c(a + b)\}}$.

(2) $\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2} = \dfrac{a^2 - b^2}{2x}$

and $(x^2 + a^2) - (x^2 + b^2) = a^2 - b^2$

$$\therefore \sqrt{x^2 + a^2} - \sqrt{x^2 + b^2} = 2x$$

$$\therefore 2\sqrt{x^2 + a^2} = 2x + \frac{a^2 - b^2}{2x}$$

$$\therefore 4x^2 + 4a^2 = 4x^2 + 2(a^2 - b^2) + \frac{(a^2 - b^2)^2}{4x^2}$$

$$\therefore 2(a^2 + b^2) = \frac{(a^2 - b^2)^2}{4x^2} \text{ \&c.}$$

(3) By subtraction we get

$$(a - b)\frac{x^2}{y^2} + (b - c)\frac{x}{y} + c - a = 0.$$

$\frac{x}{y} = 1$ evidently satisfies the equation, and the product of the roots = the last term $\therefore$ the other root is $\frac{x}{y} = \frac{c-a}{a-b}$.

If $x = y$, we have $x^2 = y^2 = \frac{d}{a+b+c}$.

If $x = \frac{c-a}{a-b}y$, $x^2 = \frac{d(a-c)^2}{a(a^2+b^2+c^2-bc-ca-ab)}$ &c.

3. $\frac{ab - r_1 r_2}{r_3} = \frac{ab - \frac{S}{s-a}\cdot\frac{S}{s-b}}{\frac{S}{s-c}} = \frac{(s-c)\{ab - s.(s-c)\}}{S} = \frac{S}{s}$ = &c.

4. $\cos\theta = \frac{a^2+b^2-c^2}{2ab}$, and $b = \frac{a+c}{2}$

$$\therefore \cos\theta = \frac{a-c}{a} + \frac{b}{2a} = \frac{5a-3c}{4a}.$$

So $\cos\phi = \frac{5c-3a}{4c}$

$$\therefore 4(1-\cos\theta)(1-\cos\phi) = \cos\theta + \cos\phi.$$

5. Let $ABC$ be a triangle whose sides are parallel to the three given straight lines, $O$ the centre of the circle, $OG$ the radius perpendicular to $BC$. Make the angle $GON$ equal to the angle $BAC$. Draw $NM$ parallel to $BC$, $MP$ parallel to $AC$. Then $MNP$ is the triangle required.

For arc $MG$ = arc $GN$, $\therefore$ angle $MOG$ = angle $GON$.

$\therefore$ angle $MPN$ = angle $BAC$ &c.

6. Let $P$, $Q$, $R$ be three points on a parabola whose ordinates are $y_1$, $y_2$, $y_3$, and let the inclinations of the tangents at $P$, $Q$, $R$ to the axis of $x$ be $\theta_1$, $\theta_2$, $\theta_3$.

Then $\cot\theta_1 + \cot\theta_3 = 2\cot\theta_2$

$$\therefore \frac{y_1}{2a} + \frac{y_3}{2a} = \frac{2y_2}{2a} \quad \therefore y_1 + y_3 = 2y_2 \quad \therefore y_3 - y_2 = y_2 - y_1 = \text{com. dif.} = \text{constant.}$$

$\therefore$ $Q$ is the vertex of the diameter $QV$ whose ordinate is $PVR$.

$\therefore$ $TQ = \frac{1}{2}TV$; and if $LQM$ the tangent at $Q$ meet $TR$ and $TP$, the tangents at $R$ and $P$, in $L$ and $M$, the triangles $LMT$, $RPT$ are similar. Also area $LMT = \frac{1}{4}RPT = \frac{1}{2}PQR = \frac{1}{2}QV.VR\sin QVR$.

But $RV^2 = QV \cdot \frac{4a}{\sin^2 QVR}$, and $RV \cdot \sin QVR = \frac{y_3 - y_1}{2}$

$$\therefore QV = \frac{(y_3 - y_1)^2}{16a}$$

$$\therefore \triangle LMT = \frac{(y_3 - y_1)^3}{64a}$$

$\therefore$ the $\triangle$ has a constant area as long as $y_3 - y_1$ is constant, which is the case in the question.

7. Let $ABCD$ be the square, $A$ resting against the wall and $B$ attached by a string to the point $E$. $BK$, $CN'$, $DH$ are perpendiculars on the wall. $G$ is the centre of the square, $T$ the tension of the string, $W$ the weight of the square, $R$ the reaction at $A$. $CL$ and $CM$ are perpendiculars on the lines of action of $T$ and $R$. If the vertical through $G$ cuts $CN'$ in $N$ and $AM$ in $F$, the line of action of $T$ must pass through $F$. Let $a$ be the angle which $AB$ makes with the wall. Then $BEA = a$.

Then $$T : W : R :: 1 : \cos a : \sin a.$$

Taking moments round $C$,

$$T \cdot CL = W \cdot CN + R \cdot CM$$

Now $CL = a \cos 2a$, $a$ being a side of the square

$$CM = CA \cos\left(\frac{\pi}{4} + a\right) = a(\cos a - \sin a)$$

$$CN = CG \sin\left(\frac{\pi}{4} + a\right) = a(\cos a + \sin a)$$

$$\therefore \cos 2a = \tfrac{1}{2} \cos a (\cos a + \sin a) + \sin a (\cos a - \sin a)$$

$$\therefore \cos^2 a = 3 \sin a \cos a, \quad \text{and} \quad a \neq \frac{\pi}{2}; \ \therefore \cos a \neq 0$$

$$\therefore \frac{\cos a}{3} = \frac{\sin a}{1};$$

$$\therefore CN' : DH : BK :: \sin a + \cos a : \cos a : \sin a$$

$$:: 4 : 3 : 1$$

PAPER III.

1. $(1-x)^{-\frac{1}{n}} = 1 + \frac{1}{n}x + \ldots + \frac{\frac{1}{n}\left(\frac{1}{n}+1\right)\ldots\left(\frac{1}{n}+r-2\right)}{r-1\,!} \cdot x^{r-1} + \ldots$

$(1-x)^{-1} = 1 + x + \ldots + x^{r-1} + \ldots$

$\therefore (1-x)^{-\left(1+\frac{1}{n}\right)} = 1 + \ldots + x^r\left\{1 + \frac{1}{n} + \ldots + \frac{\frac{1}{n}\left(\frac{1}{n}+1\right)\ldots\left(\frac{1}{n}+r-2\right)}{r-1\,!}\right\} + \ldots$

$\therefore$ required sum of coefficients = coef. $x^{r-1}$ in $(1-x)^{-\left(1+\frac{1}{n}\right)}$

$$= \frac{\left(\frac{1}{n}+1\right)\left(\frac{1}{n}+2\right)\ldots\left(\frac{1}{n}+r-1\right)}{r-1\,!}$$

$\therefore$ required ratio is $\dfrac{\frac{1}{n}+r-1}{\frac{1}{n}} = 1 + n\,(r-1) : 1$

2. $\log_e\left(\frac{1}{1-x}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$

Let $x = \frac{1}{5}$ or $\cdot 2$.

$\therefore \log_e \frac{5}{4} = \cdot 2 + \frac{(\cdot 2)^2}{2} + \frac{(\cdot 2)^3}{3} + \ldots$

$= \cdot 2 + \cdot 02 + \cdot 002\dot{6} + \cdot 0004 + \cdot 000064 + \cdot 0000106 + \ldots$

$= \cdot 2231412\ldots$

$\therefore \log_{10} \frac{5}{4} = \mu \times \cdot 2231412$ and $\frac{5}{4} = \frac{5^3}{10^2}$

$\therefore 3 \log_{10} 5 - 2 = \cdot 43429 \times \cdot 22314.$

$= \cdot 096907\ldots$

$\therefore \log_{10} 5 = \frac{1}{3})\underline{2 \cdot 096907}$

$= \cdot 698969\ldots$

$= \cdot 69897$ correct to five places of decimals.

3. $m = \dfrac{1}{\sin\theta} - \sin\theta = \dfrac{\cos^2\theta}{\sin\theta}$

So $n = \dfrac{\sin^2\theta}{\cos\theta}$ $\therefore mn = \sin\theta\cos\theta$

$$\therefore m^{\frac{2}{3}} + n^{\frac{2}{3}} = \frac{\cos^{\frac{4}{3}}\theta}{\sin^{\frac{2}{3}}\theta} + \frac{\sin^{\frac{4}{3}}\theta}{\cos^{\frac{2}{3}}\theta} = \frac{1}{\sin^{\frac{2}{3}}\theta\cos^{\frac{2}{3}}\theta} = \frac{1}{(mn)^{\frac{2}{3}}}.$$

4. For $\cos a$ write $c$.

$$\text{Let } C = 1 + \frac{\cos a}{c} + \frac{\cos 2a}{c^2} + \ldots + \frac{\cos(n-1)a}{c^{n-1}}$$

$$S = \frac{\sin a}{c} + \frac{\sin 2a}{c^2} + \ldots + \frac{\sin(n-1)a}{c^{n-1}}$$

$$\therefore C + Si = 1 + \frac{e^{ai}}{c} + \frac{e^{2ai}}{c^2} + \ldots + \frac{e^{(n-1)ai}}{c^{n-1}}$$

$$= \frac{1 - \dfrac{e^{nai}}{c^n}}{1 - \dfrac{e^{ai}}{c}}$$

$$\therefore (C + Si)\left(1 - \frac{e^{ai}}{c}\right) = 1 - \frac{e^{nai}}{c^n}$$

$$\therefore (C + Si)\left\{1 - \frac{1}{c}(\cos a + i\sin a)\right\} = 1 - \frac{1}{c^n}(\cos na + i\sin na)$$

$\therefore$ equating unreal parts

$$C.\frac{\sin a}{c} - S\left(1 - \frac{\cos a}{c}\right) = \frac{\sin na}{c^n}$$

and $c = \cos a$ $\therefore C = \dfrac{\sin na}{\cos^n a} \cdot \cot a = 0$ if $na = \pi$.

By equating real parts we find

$$-S\sin a = 1 - \frac{\cos na}{\cos^n a}.$$

5. Let $O$ be the centre, $R$ the radius of the circle round $ABCD$, and let $G$ be the other point of intersection of the circles round $FDC$, $BCE$. Then the angle $FGC = ADC = CBE$.

$\therefore EGC + FGC = EGC + EBC =$ two right angles.

$\therefore EG$ and $GF$ are in one straight line.

Bisect $EF$ in $L$, and draw the tangents $FH$, $EK$, $LM$ to the circle round $ABCD$.

Then $$FG \cdot EF = FC \cdot FB = FH^2$$

and $$EG \cdot EF = EC \cdot ED = EK^2$$

$$\therefore EF^2 = FH^2 + EK^2$$

$$\therefore 4FL^2 + 2R^2 = FH^2 + R^2 + EK^2 + R^2$$

$$= FO^2 + EO^2$$

$$= 2OL^2 + 2FL^2$$

$$\therefore FL^2 + R^2 = OL^2 = LM^2 + R^2$$

$$\therefore FL = LM.$$

$\therefore$ the circle, centre $L$ and radius $LE$ or $LF$ passes through $M$. And the radii $LM$, $MO$ are at right angles, $\therefore$ the circles cut orthogonally.

6. Let the forces be denoted by $\frac{\mu}{AB}$, $\frac{\mu}{BC}$.

Then resolving along the normal at $B$ we have

$$\mu \cdot \frac{\sin C}{c} - \mu \cdot \frac{\sin A}{a}, \text{ which } = 0.$$

$\therefore$ the resultant must act along the tangent.

7. Let the diameter through $Q'$ meet the tangent at $P$ in $T$. Join $RQ'$ cutting the curve in $L$ and the diameter at $P$ in $M$.

Then $$RL \cdot RQ' = RM^2.$$

Now $$RM : RQ' :: RP : RT :: 1 : 3 \therefore RM = \tfrac{1}{3} RQ'.$$

$$\therefore RL \cdot RQ' = \tfrac{1}{9} RQ'^2 \therefore RL = \tfrac{1}{9} RQ'.$$

$$\therefore RL : LQ' :: 1 : 8.$$

Similarly $$R'L' : L'Q :: 1 : 8.$$

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## Paper IV.

1. $$2a - 3y = \frac{(z-x)^2}{y}; \quad 2a - 3z = \frac{(x-y)^2}{z}$$

$$\therefore 2ay - 3y^2 = z^2 + x^2 - 2zx$$

$$2az - 3z^2 = y^2 + x^2 - 2xy$$

$$\therefore (y - z)\{2a - 3(y + z)\} = (y - z)\{2x - (y + z)\}$$

and $$y - z \neq 0 \therefore x + y + z = a.$$

Again $$3(y - z) = \frac{(x - y)^2}{z} - \frac{(z - x)^2}{y}$$

$$= \frac{y(x^2 + y^2 - 2xy) - z(z^2 + x^2 - 2zx)}{yz}$$

$$= \frac{x^2(y - z) + (y - z)(y^2 + z^2 + yz) - 2x(y^2 - z^2)}{yz}$$

$$\therefore 3yz = x^2 + y^2 + z^2 + yz - 2xy - 2xz$$

$$\therefore (y - z)^2 = x(2y + 2z - x)$$

$$= x(2a - 3x)$$

$$\therefore 2a - 3x = \frac{(y - z)^2}{x}.$$

2. The given expression

$$= 2\cos\frac{5a - 2\beta - \gamma}{4}\left\{\cos\frac{4\beta + 3\gamma - 3a}{4} + \cos\frac{6\beta - 7\gamma + a}{4}\right\}$$

$$= \cos\frac{2\beta + 2\gamma + 2a}{4} + \cos\frac{8a - 6\beta - 4\gamma}{4} + \cos\frac{6a + 4\beta - 8\gamma}{4} + \cos\frac{4a - 8\beta + 6\gamma}{4}$$

= required result, since $\alpha + \beta + \gamma = \pi$.

3. Let $ABC$ be the given triangle, $O_2$, $O_3$ the centres of the escribed circles opposite $B$ and $C$. $O_2D$, $O_2E$ are perpendiculars on $BA$, $BC$ and $O_3F$, $O_3G$ are perpendiculars on $CA$, $CB$. $ED$, $GF$ are produced to meet in $A'$.

Then the angle $BDE = BO_2E = 90° - \frac{B}{2}$

$$\therefore ADA' = 90° + \frac{B}{2}. \quad \text{So } AFA' = 90° + \frac{C}{2}$$

$$\therefore A' = 360° - (180° + \frac{B + C}{2} + A) = 90° - \frac{A}{2}$$

So $A'' = 90° - \frac{A'}{2} = 45° + \frac{A}{4}$; $B'' = 45° + \frac{B}{4}$; $C'' = 45° + \frac{C}{4}$

$A''' = 90° - \frac{A''}{2} = 67\frac{1}{2}° + \frac{A}{8}$, $B''' = 67\frac{1}{2}° + \frac{B}{8}$; $C''' = 67\frac{1}{2}° + \frac{C}{8}$

. . . . . .

$$A_n = D + \frac{A}{2^n}, \; B_n = D + \frac{B}{2^n}, \; C_n = D + \frac{C}{2^n}.$$

Now when $n$ is indefinitely increased, each of the angles $\frac{A}{2n}, \frac{B}{2n}, \frac{C}{2n}$ ultimately vanishes, and $A_n$, $B_n$, $C_n$ each become $= D$, which is $\therefore 60°$.

$\therefore$ the triangles are ultimately equilateral.

4. Let the forces be denoted by $P$, $Q$, $R$. Their directions pass through $O$ the centre of the circum-circle

$$\therefore P : Q : R :: BC : CA : AB$$
$$:: \sin A : \sin B : \sin C$$
$$:: \sin EOF : \sin FOD : \sin DOE$$

$\therefore$ the forces are in equilibrium.

5. It is true that one of the values of $e^{2n\pi\sqrt{-1}}$ is

$$\cos 2n\pi + \sqrt{-1} \sin 2n\pi$$

and that one of the values of each of these expressions is unity,

$\therefore$ one of the values of $e^{2\pi\sqrt{-1}}$ = one of the values of $e^{4\pi\sqrt{-1}}$.

To find the other values of the expression $e^{2n\pi\sqrt{-1}}$, introduce the common factor $x$, and let the arithmetic logarithm of $x$ be $a$.

$$\therefore x \,.\, e^{2n\pi\sqrt{-1}} = x$$
$$\log x = a + 2n\pi\sqrt{-1}.$$

Thus we see that a quantity has an infinite number of logs, one being real and the rest imaginary, and the latter are found by adding to the real log the quantity $2n\pi\sqrt{-1}$. The suitable value to be given to $n$ is to be determined from the circumstances of each case.

Thus to be absolutely correct, in the question the equality

$$e^{2\pi\sqrt{-1}} = e^{4\pi\sqrt{-1}}$$

should be written

$$e^{2\pi\sqrt{-1}} = e^{4\pi\sqrt{-1}} \,.\, e^{2n\pi\sqrt{-1}}$$

$\therefore$ raising each to power $\sqrt{-1}$

$$e^{-2\pi} = e^{-4\pi}\, e^{-2n\pi}$$

and this is evidently true when $n = -1$.

Similarly it can be shewn that there is a value of each of the other expressions equal to one of the values of $e^{2\pi\sqrt{-1}}$.

6. Let $A$ and $B$ be the centres of the circles which intersect in $C$ and $G$, and let the tangents at $C$ and the common chord cut $AB$ in $D$, $E$, $F$ respectively, so that $A$ and $D$ are on the same side of the common chord $CFG$.

Then $a = DCE = DCG + ECG = CBA + CAB = \pi - ACB$

$$\therefore \frac{BC}{AB} = \frac{\sin CAB}{\sin ACB} = \frac{CF}{CA} \cdot \frac{1}{\sin a}$$

$$\therefore CF = \frac{ab \sin a}{AB} = \frac{ab \sin a}{\sqrt{a^2 + b^2 + 2ab \cos a}}.$$

7. Draw $RM$ perpendicular to $SH$.

Since $CR = CS = CH$ $\therefore$ $SRH$ is a right angle.

Since $CS = CR$ $\therefore$ $CSR = CRS = RSY$

$\therefore$ the triangle $SRM$ = triangle $SRY$ $\therefore$ $SM = SY$.

Similarly it can be shewn that $HM = HZ$.

Since $RM^2 = SM \,.\, MH = SY \,.\, HZ = BC^2$ $\therefore$ $RM = BC$, $\therefore$ $R$ lies on the tangent at $B$.

Again
$$SP : HP :: SY : HZ$$
$$:: SM : HM$$
$$:: SM \,.\, HM : HM^2$$
$$:: RM^2 : HM^2$$
$$:: SR^2 : HR^2.$$

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PAPER V.

1.

$$\begin{array}{rrr} & & 1037\ (10{\cdot}12 \\ & & 1 \\ \hline 301 & 30000 & 37000 \\ 2{\cdot} & 301 & \\ \hline & 30301 & 30301 \\ \hline 3032 & 3090300 & 6699000 \\ & 6064 & \\ \hline & 3096364 & \end{array}$$

2. Suppose that the expressions are equal when $a$ and $b$ are interchanged.

Then
$$0 = a + \frac{bc - a^2}{a^2 + b^2 + c^2} - \left(b + \frac{ca - b^2}{a^2 + b^2 + c^2}\right)$$
$$= a - b + \frac{bc - ca - (a^2 - b^2)}{a^2 + b^2 + c^2}$$
$$= (a - b)\left\{1 - \frac{a + b + c}{a^2 + b^2 + c^2}\right\}$$
$$\therefore 1 - \frac{a + b + c}{a^2 + b^2 + c^2} = 0 \text{ since } a - b \neq 0.$$

Now we will prove that when this condition holds, the expressions are equal when $a$ and $c$ are interchanged.

For
$$a + \frac{bc - a^2}{a^2 + b^2 + c^2} - \left(c + \frac{ab - c^2}{a^2 + b^2 + c^2}\right)$$
$$= a - c + \frac{bc - ab - (a^2 - c^2)}{a^2 + b^2 + c^2}$$
$$= (a - c)\left\{1 - \frac{a + b + c}{a^2 + b^2 + c^2}\right\}$$
$$= 0.$$

Also, if $a + b + c = 1$, we have $a^2 + b^2 + c^2 = 1$,
$$\therefore (b + c)^2 = (1 - a)^2$$
$$\therefore b^2 + c^2 + 2bc = 1 - 2a + a^2,$$
$$\therefore 1 + 2bc - a^2 = 1 - 2a + a^2,$$
$$\therefore bc + a - a^2 = 0,$$
$$\therefore a + \frac{bc - a^2}{a^2 + b^2 + c^2} = a + bc - a^2 = 0.$$

3. Assume $y = x^2$, so that when $x$ has any particular value, $y =$ the square of that value. $\therefore x = \sqrt{y}$, and the required equation is $ay + b\sqrt{y} + c = 0$, which reduces to
$$a^2y^2 + (2ac - b^2)y + c^2 = 0.$$

Again, assume $y = \sqrt{x}$, so that when $x$ has any particular value, $y =$ the square root of that value. $\therefore x = y^2$, and the required equation is $ay^4 + by^2 + c = 0$.

4. $\cos 2\theta - \cos 4\theta = 2\cos 2\theta \cos 4\theta$

$= \cos 2\theta + \cos 6\theta,$

$\therefore \cos 6\theta = -\cos 4\theta = \cos(\pi - 4\theta)$

$\therefore 6\theta = 2n\pi \pm (\pi - 4\theta)$

$\therefore$ either $10\theta = (2n+1)\pi$, or $2\theta = (2n-1)\pi$.

5. If $h$ denote the height of the room

$$h^2 \cot^2\alpha = AC^2 = a^2 + BC^2 = a^2 + h^2 \cot^2\beta,$$

$$\therefore h^2 = \frac{a^2}{\cot^2\alpha - \cot^2\beta}.$$

Now if $\alpha = 18°$, $\cot^2\alpha = 5 + 2\sqrt{5}$, and if $\beta = 30°$, $\cot^2\beta = 3$,

$$\therefore h^2 = \frac{48^2}{5 + 2\sqrt{5} - 3} = \frac{48^2}{2(1+\sqrt{5})} = \frac{48^2}{4^2}(2\sqrt{5}-2) = 12^2(2\sqrt{5}-2)$$

$= 12^2 \times 2{\cdot}47212 \ldots .$

$\therefore h = 12 \times 1{\cdot}572$ ft.

$= 18$ ft. 10 in. ·368, &c.

6. Let $\theta$ be the eccentric angle of $P$. The equation of any one of the series of parabolas satisfying the given conditions is

$$(x - a\cos\theta)^2 + (y - b\sin\theta)^2 = (x\cos\alpha + y\sin\alpha - p)^2.$$

If $y = 0$, $x = \pm\sqrt{a^2 - b^2}$. This gives us

$$\therefore a\cos\theta = p\cos\alpha$$

and $(a^2 - b^2)\sin^2\alpha = p^2 - a^2\cos^2\theta - b^2\sin^2\theta,$

$\therefore$ eliminating $p$ we have

$$(a^2\sin^2\alpha + b^2\cos^2\alpha)(\sin^2\theta - \sin^2\alpha) = 0, \quad \therefore \alpha = n\pi \pm \theta.$$

By giving different values to $n$, it will be found that there are only two different parabolas. Their equations are

$$(x - a\cos\theta)^2 + (y - b\sin\theta)^2 = (x\cos\theta + y\sin\theta - a)^2$$

$$(x - a\cos\theta)^2 + (y - b\sin\theta)^2 = (x\cos\theta - y\sin\theta - a)^2.$$

The equations to the directrices are

$$y = -x\cot\theta + a\operatorname{cosec}\theta, \; y = x\cot\theta + a\operatorname{cosec}\theta,$$

and these make with the axis of $x$ angles $\frac{\pi}{2} + \theta$ and $\frac{\pi}{2} - \theta$,

$\therefore$ the angle between them is $2\theta$.

7. $CD^2 = AC \cdot AD = AC(AC - CD)$,

$$\therefore CD^2 + AC \cdot CD - AC^2 = 0.$$

$$\therefore CD = \frac{-1 \pm \sqrt{5}}{2} AC.$$

$\therefore$ taking the positive sign we get

$$BE = BD = BC + CD = AC + \frac{\sqrt{5}-1}{2} AC = \frac{\sqrt{5}+1}{2} AC$$

$$\therefore \cos ABE = \frac{BE}{AB} = \frac{BE}{2AC} = \frac{\sqrt{5}+1}{4} = \cos 36°$$

$\therefore ACE = 72° = \frac{1}{5}$ of four right angles.

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## Paper VI.

1. Substitute for $y$ in terms of $x$ from (1) in (2) and we get

$$x^2(b^2c + a^2d) - 2xad + d - b^2 = 0.$$

If this has equal roots

$$a^2d^2 = (b^2c + a^2d)(d - b^2), \text{ which reduces to}$$

$$a^2d + b^2c = cd, \text{ or } \frac{a^2}{c} + \frac{b^2}{d} = 1.$$

Now $x = \dfrac{ad}{b^2c + a^2d} = \dfrac{ad}{dc} = \dfrac{a}{c}$

$$y = \frac{1}{b}(1 - ax) = \frac{c - a^2}{bc} = \frac{cd - a^2d}{bcd} = \frac{b^2c}{bcd} = \frac{b}{d}.$$

2. $8^a = 9 \therefore 2^{3a} = 3^2$; $3^b = 5 \therefore 3 = 5^{\frac{1}{b}}$

(1) Let $\log_{10}1 = x_1 \therefore 10^{x_1} = 1 \therefore x_1 = 0$

$\log_{10}2 = x_2 \therefore 10^{x_2} = 2.$

Then $2 = 3^{\frac{2}{3a}} = 5^{\frac{2}{3ab}} = \left(\frac{10}{2}\right)^{\frac{2}{3ab}}$

$$\therefore 2^{1+\frac{2}{3ab}} = 10^{\frac{2}{3ab}} \therefore 2 = 10^{\frac{2}{2+3ab}}$$

$$\because 10^{x_2} = 10^{\frac{2}{2+3ab}} \therefore x_2 = \frac{2}{2+3ab}$$

(3) $$\log_{10}3 = x_3$$

$$\therefore 10^{x_3} = 3 = 2^{\frac{3a}{2}} = 10^{\frac{3a}{2+3ab}} \therefore x_3 = \frac{3a}{2+3ab}.$$

(4) $$\log_{10}4 = x_4$$

$$\therefore 10^{x_4} = 4 = 2^2 = 10^{\frac{4}{2+3ab}} \therefore x_4 = \frac{4}{2+3ab}.$$

3. $$\sin B = \sin(C + A)$$

$$\therefore \sin^2 B + \sin^2 C - \sin^2 A$$
$$= \sin B \,.\, \sin(C + A) + \sin(C + A)\sin(C - A)$$
$$= \sin B \left\{\sin(C + A) + \sin(C - A)\right\}$$
$$= 2 \sin B \sin C \cos A.$$

4. Let $P$ be the orthocentre of $ABC$.

Then $$PDF = PBF = \frac{\pi}{2} - A$$

$$PDE = PCE = \frac{\pi}{2} - A$$

$\therefore PD$ bisects the angle $EDF$. $\therefore P$ is the centre of the circle inscribed in $DEF$. Let $r'$ denote its radius.

Then $$FE = AE \,.\, \frac{\sin A}{\sin C} = c \cos A \,.\, \frac{a}{c} = a \cos A$$

$$\therefore FE = r'\left(\cot\frac{DEF}{2} + \cot\frac{EFD}{2}\right). \quad \textit{Todh. Trig.} \text{ Art. 249.}$$

$$\therefore a \cos A = r'(\tan B + \tan C) = r' \frac{\sin(B + C)}{\cos B \cos C} = r' \frac{\sin A}{\cos B \cos C}$$

$$\therefore r' = \frac{a}{\sin A} \cos A \cos B \cos C$$
$$= 2R \cos A \cos B \cos C.$$

5. Let $AG$ be the vertical wall, $ABCD$ the rod, $C$ being its middle point and $B$ the position of the rail. Draw $BG$ and $AF$ perpendicular to the wall, and $CF$ vertical. Then $P$, the reaction at $B$ passes through $F$. Let $W$ be the weight of the rod, and $\theta$ the angle it makes with the horizon, and $a$ its length. Taking moments about $A$ we have

$$P \,.\, AB = W \,.\, AF$$

$$AB = \frac{BG}{\cos\theta} = \frac{a}{16} \cdot \frac{1}{\cos\theta}; \; AF = AC\cos\theta = \frac{a}{2}\cos\theta$$

$$\therefore P = W \cdot 8\cos^2\theta.$$

Also $$P : W :: \sin AFC : \sin AFP$$
$$:: 1 : \cos\theta.$$

$$\therefore P = W \cdot \frac{1}{\cos\theta} = W \cdot 8\cos^2\theta$$

$$\therefore \cos^3\theta = \tfrac{1}{8} \therefore \cos\theta = \tfrac{1}{2} \therefore \theta = 60°.$$

6. Join $DB$, $EG$, $FH$, $FE$, $HG$.

Then $AE : EB :: AG : GD$ $\therefore$ $GE$ is parallel to $DB$. Similarly $FH$ is parallel to $DB$ $\therefore$ $GE$ is parallel to $FH$ $\therefore$ they are in the same plane $\therefore$ $GH$ and $EF$ are in the same plane with them.

7. Tripos 1878. Monday morning. No. 11.

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## Paper VII.

1. Clear of fractions, and we get

$$x^3(a+b-c-d) + 2x^2(ab-cd) + x\{ab(c+d) - cd(a+b)\} = 0.$$

Then $x = 0$ is one root. Since two of the roots are equal, suppose

(1) the equal roots to be $= 0$.

$$\therefore ab(c+d) = cd(a+b)$$
$$\therefore \frac{1}{a} + \frac{1}{b} = \frac{1}{c} + \frac{1}{d}.$$

(2) If the equal roots be not $= 0$

$$x^2(a+b-c-d) + 2x(ab-cd) + ab(c+d) - cd(a+b) = 0.$$

Since this equation has equal roots

$$\therefore (ab-cd)^2 - (a+b-c-d)\{ab(c+d) - cd(a+b)\} = 0.$$

Now the expression on the left vanishes when $a = c$, $a = d$, $b = c$, or $b = d$, and it is of the fourth degree, $\therefore$ we may put it

$\equiv k(a-c)(a-d)(b-c)(b-d)$ where $k$ is some numerical constant.

Let $a = b = 2$, $c = d = 1$ $\therefore k = 9 - 2(8-4) = 1$ $\therefore k = 1$

$\therefore$ one of the quantities $a$ or $b$ is equal to one of the quantities $c$ or $d$.

If $b = d$, $x = -\dfrac{ab - cd}{a + b - c - d} = -\dfrac{ab - cb}{a - c} = -b, -b, 0.$

If $a = c$, $x = -\dfrac{ab - cd}{a + b - c - d} = -\dfrac{ab - ad}{b - d} = -a, -a, 0.$

2. $(1 + 1)^{\frac{3}{2}} = 1 + \frac{3}{2} + \dfrac{\frac{3}{2}(\frac{3}{2} - 1)}{2!} + \dfrac{\frac{3}{2}(\frac{3}{2} - 1)(\frac{3}{2} - 2)}{3!} + \ldots$

$= 1 + \dfrac{3}{2} + \dfrac{3.1}{2^2.2!} - \dfrac{3.1.1}{2^3.3!} + \dfrac{3.1.1.3}{2^4.4!} - \dfrac{3.1.1.3.5}{2^5.5!} + \ldots.$

$= 1 + \dfrac{3}{2} + \dfrac{3\,1}{2^2.2!} - 3\left\{\dfrac{1}{2^3.3!} - \dfrac{1.3}{2^4.4!} + \dfrac{1.3.5}{2^5.5!} \ldots\right\}$

$\therefore \dfrac{1}{2^3.3!} - \dfrac{1.3}{2^4.4!} + \dfrac{1\,3.5}{2^5.5!} - \ldots.$

$= \dfrac{1}{3}\left\{1 + \dfrac{3}{2} + \dfrac{3.1}{2^2.2!} - 2^{\frac{3}{2}}\right\}$

$= \dfrac{1}{3}\left(\dfrac{23}{8} - 2\sqrt{2}\right) = \dfrac{23}{24} - \dfrac{2}{3}\sqrt{2}.$

3. $\left.\begin{aligned} x - y\cos R - z\cos Q &= 0 \\ x\cos R - y + z\cos P &= 0 \end{aligned}\right\}$

$\therefore$ since $P + Q + R = (2n + 1)\pi$, we get

$$\frac{x}{\sin P} = \frac{y}{\sin Q} = \frac{z}{\sin R} = K \text{ suppose}$$

$\therefore x\cos Q + y\cos P - z = K\{\sin P\cos Q + \cos P\sin Q - \sin R\}$

$= K\{\sin(P + Q) - \sin R\}$

$= 0.$

To find $\cos P$, multiply (1) by $x$, (2) by $y$, (3) by $z$, and subtract the first result from the sum of the two latter. We thus obtain

$$y^2 + z^2 - x^2 = 2yz\cos P.$$

4. Let $P$ be the orthocentre, $O$ and $Q$ the centres of the circum and nine-point circles. Then $Q$ bisects $OP$. From $O$, $Q$, $P$ draw $OM$ $QR$, $PN$ perpendiculars to $BC$.

Then $PN = PB\cos C = \dfrac{c}{\sin C}\cos B\cos C = 2R\cos B\cos C,$

and $OM = R\cos A.$

$$\therefore\ QR = \tfrac{1}{2}(PN + OM) = \frac{R}{2}(2\cos B\cos C + \cos A)$$
$$= \frac{R}{2}\cos(C - B).$$

5. Tripos 1875, Monday morning. No. 1.

6. Substitute and clear of fractions.

$$\therefore\ ax + b + \sqrt{-1}\,ay = (cx + d)\xi - cy\eta + \sqrt{-1}\{(cx + d)\eta + cy\xi\}.$$

Equate real and unreal parts.

$$\left.\begin{aligned} \therefore\ x(a - c\xi) + c\eta y &= d\xi - b \\ x.c\eta - y(a - c\xi) &= -d\eta \end{aligned}\right\}$$

$$\therefore\ x\{(a - c\xi)^2 + c^2\eta^2\} = (a - c\xi)(d\xi - b) - cd\eta^2,$$

$$y\{(a - c\xi)^2 + c^2\eta^2\} = c\eta(d\xi - b) + d\eta(a - c\xi),$$

$$\therefore\ (x^2 + y^2)\{(a - c\xi)^2 + c^2\eta^2\}^2 = (a - c\xi)^2(d\xi - b)^2 + c^2d^2\eta^2 + c^2\eta^2(d\xi - b)^2 + d^2\eta^2(a - c\xi)^2,$$
$$= \{(a - c\xi)^2 + c^2\eta^2\}\{(d\xi - b)^2 + d^2\eta^2\},$$

$$\therefore\ (x^2 + y^2)\{(a - c\xi)^2 + c^2\eta^2\} = (d\xi - b)^2 + a^2\eta^2.$$

Now if $x, y$ describe a circle, we must have

$$k^2 = x^2 + y^2 + px + qy, \text{ where } k \text{ is constant,}$$

$$\therefore\ k^2\{(a - c\xi)^2 + c^2\eta^2\} = \{(a - c\xi)^2 + c^2\eta^2\}\{x^2 + y^2 + px + qy\},$$
$$= (d\xi - b)^2 + d^2\eta^2 + p(a - c\xi)(d\xi - b) - pcd\eta^2$$
$$+ qc\eta(d\xi - b) + qd\eta(a - c\xi).$$

On simplifying, this reduces to the form

$$(\xi^2 + \eta^2)(c^2k^2 - d^2 + pcd) + A\xi + B\eta + C = 0$$

where $A$, $B$, and $C$ do not involve $\xi$ or $\eta$.

$\therefore$ the locus of $\xi, \eta$ is a circle.

7. Let $PM$, $QON$ be the ordinates at $P$ and $O$. Draw $SY$ perpendicular to the tangent at $P$.

Then $NG = 2AS$, $QN = PM = 2AY$.

$\therefore\ QN : NG :: YA : AS$, $\therefore$ $QG$ is parallel to $SY$.

## Paper VIII.

1. Referring to Paper VI. 2, we see that

$$10x_4 = 4 = 2^2 = 10^{\frac{4}{2+3ab}} \quad \therefore x_4 = \frac{4}{2+3ab};$$

$$10x_5 = 5 = 3^b = 2^{\frac{3ab}{2}} \quad \therefore x_5 = \frac{3ab}{2+3ab};$$

$$10x_6 = 6 = 2\cdot 3 = 2^{\frac{2+3a}{2}} \quad \therefore x_6 = \frac{2+3a}{2+3ab};$$

$$10x_7 = 7 = 5^c = 10^{\frac{3abc}{2+3ab}} \quad \therefore x_7 = \frac{3abc}{2+3ab};$$

$$10x_8 = 8 = 2^3 = 10^{\frac{6}{2+3ab}} \quad \therefore x_8 = \frac{6}{2+3ab};$$

$$10x_9 = 9 = 3^2 = 2^{3a} = 10^{\frac{6a}{2+3ab}} \quad \therefore x_9 = \frac{6a}{2+3ab}.$$

2. If $x = 1$, the expression $= 1$.

Take any two consecutive values of $x$ and add them, and we get $\frac{1}{5^5}(3\,.\,10^x + 3\,.\,10^{x+1})$, which $= \frac{3}{5^5}\,.\,10^x(10+1)$, which is always a positive integer. Thus the sum of the numbers formed by giving consecutive values to $x$ is a positive integer. Now we know that the number obtained by putting $x = 1$ is a positive integer, viz. 1, $\therefore$ the 2nd is a positive integer, and so on.

3. Let $O$, $O_1$ be the centres of the circles inscribed in $ABC$ and $AEOF$. Then $O_1$ lies on $AO$. Draw $O_1E_1$, $O_1H$ perpendicular to $AB$ and $OE$ respectively. Then since $O_1E$ bisects the angle $OEA$,

$\therefore O_1EE_1 = 45° = EO_1E_1 \quad \therefore O_1H = EE_1 = OE_1 = r_1$, and $r - r_1 = OH$,

$$\therefore \frac{r_1}{r-r_1} = \frac{O_1H}{OH} = \cot\frac{A}{2}. \quad \text{So } \frac{r_2}{r-r_2} = \cot\frac{B}{2}, \ \frac{r_3}{r-r_3} = \cot\frac{C}{2}.$$

But $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

$$\therefore \frac{r_1}{r-r_1} + \frac{r_2}{r-r_2} + \frac{r_3}{r-r_3} = \frac{r_1}{r-r_1}\cdot\frac{r_2}{r-r_2}\cdot\frac{r_3}{r-r_3}.$$

4. The angle $M = \pi - \frac{1}{2}(A+D)$; the angle $P = \pi - \frac{1}{2}(B+C)$;

$\therefore M + P = 2\pi - \frac{1}{2}(A+B+C+D) = 2\pi - \pi = \pi$.

5. Since $YP$, $YP'$ are tangents from $Y$, $\therefore CY$ bisects $PP'$. Also, since $C$ is the middle point of $P'Q$, $\therefore PQ$ is parallel to $CY$, and is $\therefore$ a normal at $P$.

6. Taking moments about $A$ and $C$

$$\gamma \,.\, CD \,.\, AD \,.\, \sin D = -\beta \,.\, BC \,.\, AB \,.\, \sin B; \quad (1)$$

$$\therefore -\beta \,.\, \Delta ABC = \gamma \,.\, \Delta ACD = \gamma(ABCD - \Delta ABC).$$

$$\therefore (\gamma - \beta)\Delta ABC = \gamma \,.\, ABCD;$$

$$\text{So } \alpha \,.\, AB \,.\, CD \,.\, \sin B = -\delta \,.\, AD \,.\, CD \,.\, \sin D; \quad (2)$$

$\therefore$ from (1) and (2) $\alpha\gamma = \beta\delta$. Taking moments about $D$,

$$\alpha \,.\, AB \,.\, AD \sin A = -\beta \,.\, BC \,.\, CD \,.\, \sin C;$$

$$\therefore \alpha \,.\, \Delta ABD = -\beta \,.\, \Delta BCD = -\beta(ABCD - \Delta ABD),$$

$\therefore (\beta - \alpha)\Delta ABD = \beta \,.\, ABCD$. And $(\gamma - \beta) \,.\, \Delta ABC = \gamma \,.\, ABCD$.

$$\therefore \frac{\Delta ABD}{\Delta ABC} = \frac{\beta}{\gamma} \cdot \frac{\gamma - \beta}{\beta - \alpha} = \frac{\alpha}{\delta} \cdot \frac{\gamma - \beta}{\beta - \alpha}.$$

7. The equation of the perpendiculars is

$$\frac{kx - hy + a(y - k)}{\sqrt{k^2 + (h - a)^2}} = \pm \frac{(kx - hy) - a(y - k)}{\sqrt{k^2 + (h + a)^2}}.$$

Square both sides, and clear of fractions;

$$\left\{k^2 + h^2 + a^2 + 2ah\right\}\left\{(kx - hy)^2 + a^2(y - k)^2 + 2a(kx - hy)(y - k)\right\}$$

$$= \left\{k^2 + h^2 + a^2 - 2ah\right\}\left\{(kx - hy)^2 + a^2(y - k)^2 - 2a(kx - hy)(y - k)\right\},$$

$$\therefore 4ah\left\{(kx - hy)^2 + a^2(y - k)^2\right\} + 4a(k^2 + h^2 + a^2)(kx - hy)(y - k) = 0$$

$$\therefore h\left\{k(x - h) - h(y - k)\right\}^2 + ha^2(y - k)^2 + (k^2 + h^2 + a^2)(y - k)$$

$$\left\{k(x - h) - h(y - k)\right\} = 0$$

$$\therefore hk^2(x - h)^2 - 2h^2k(y - k)(x - h) + h^3(y - k)^2 + ha^2(y - k)^2$$

$$- (hk^2 - h^3 - ha^2)(y - k)^2 + k(k^2 + h^2 + a^2)(x - h)(y - k) = 0$$

$$\therefore hk\left\{(x - h)^2 - (y - k)^2\right\} = (h^2 - k^2 - a^2)(x - h)(y - k).$$

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## Paper IX.

1. Let $cx^2 - ax + b$ be one factor of $ax^3 - bx^2 + c$, and let $kx + m$ be the other,

$$\therefore ax^3 - bx^2 + c = (cx^2 - ax + b)(kx + m),$$

$$= kcx^3 + x^2(cm - ak) + x(bk - am) + bm.$$

$\therefore$ equating coefficients of like powers of $x$,

$$\frac{k}{a} = \frac{m}{b} = \frac{cm - ak}{bc - a^2} = \frac{-b}{bc - a^2} = \frac{kc}{ac} = \frac{a}{ac} = \frac{1}{c} \qquad \therefore -bc = bc - a^2,$$

$$\therefore a^2 = 2bc,\ k = \frac{a}{c},\ m = \frac{c}{b},$$

$$\therefore (cx^2 - ax + b)\left(\frac{x}{m} + \frac{k}{m}\right) = \frac{c}{m}x^3 + x^2\left(\frac{ck}{m} - \frac{a}{m}\right) + x\left(\frac{b}{m} - \frac{ak}{m}\right) + \frac{bk}{m},$$

$$= bx^3 - cx + a.$$

2. The $(r-1)^{\text{th}}$ term is

$$\frac{r^2}{\left\{1^2 + 2^2 + \ldots + (r-1)^2\right\}\left\{1^2 + 2^2 + \ldots + r^2\right\}}$$

$$= \frac{r^2}{\frac{(r-1)r(2r-1)}{6} \cdot \frac{r(r+1)(2r+1)}{6}},$$

$$= \frac{36}{(r-1)(r+1)(2r-1)(2r+1)},$$

$$= \frac{A}{r-1} + \frac{B}{r+1} + \frac{C}{2r-1} + \frac{D}{2r+1},$$

$$= \frac{6}{r-1} - \frac{6}{r+1} - \frac{24}{2r-1} + \frac{24}{2r+1}, \text{ by partial fractions.}$$

$\therefore$ if $S_n$ denote the sum of $n$ terms,

$$S_n = \frac{6}{2-1} + \frac{6}{3-1} - \frac{6}{n+1} - \frac{6}{n+2} - \frac{24}{4-1} + \frac{24}{2n+3},$$

$$= 1 - \frac{6}{(n+1)(n+2)(2n+3)}.$$

3. $p\sin B + q\cos B = r(\sin B + \cos B)$,

$$\therefore (p-r)\sin B = (r-q)\cos B,$$

$$\therefore \frac{\sin B}{r-q} = \frac{\cos B}{p-r} = \frac{1}{\sqrt{(p-r)^2 + (q-r)^2}}.$$

Now $1 = \sin^2 A + \cos^2 A = p^2\sin^2 B + q^2\cos^2 B$,

$$\therefore (q-r)^2 + (p-r)^2 = p^2(q-r)^2 + q^2(p-r)^2,$$

$$\therefore (p-r)^2(1-q^2) + (q-r)^2(1-p^2) = 0.$$

4. Let $D$ be the middle point of $BC$. Then $BOD = A$,

$$\therefore p = OD = BD\cot A = \frac{a}{2}\cot A. \quad \text{So } q = \frac{b}{2}\cot B,\ r = \frac{c}{2}\cot C.$$

$$\therefore \frac{qr}{bc} + \frac{rp}{ca} + \frac{pq}{ca} = \tfrac{1}{4}(\cot B \cot C + \cot C \cot A + \cot A \cot B),$$

$$= \tfrac{1}{4} \frac{\tan A + \tan B + \tan C}{\tan A \tan B \tan C},$$

$$= \tfrac{1}{4}.$$

5. Let $EH$, $FH$ bisect the angles at $E$ and $F$, and let $EH$ meet $BC$ in $L$ and $AD$ in $K$. Then since the bisectors at $E$ are parallel to those at $F$, $\therefore$ $EH$ is perpendicular to $HF$.

$\therefore$ $FAB = AKE + AEK = FLK + KEC = CLE + LEC = BCD$.

$\therefore$ a circle will go round $ABCD$.

Let the bisector of $CGD$ meet $AB$, $HE$, $DC$ in $P$, $R$, $Q$, respectively.

Then $EPG = BAG + PGA = BDC + DGQ = CQG$.

And $PER = QER$, $\therefore$ $PQ$ is at right angles to $ER$, and is $\therefore$ parallel to $HF$, $\therefore$ the other bisector of $G$ is parallel to $HE$.

6. Let $TP$, $TQ$ be the tangents, $PQ$ normal at $P$, and let $pq$ be the tangent parallel to $PQ$. Then $pq$ bisects $TP$ and $TQ$.

Since $TPSq$ lie on a circle,

$\therefore$ $TSq = Tpq = TPQ =$ a right angle.

NOTE.—The point $p$ evidently lies on the directrix, and $T$ and $P$ are equidistant from the directrix.

7. Let $2a$ be the angle between the lines, and take the bisectors of the angles between them as axes. Then if $(h, 0)$ be any point on the axis of $x$, the equation of the circle centre $(h, 0)$ touching the lines is

$$(x - h)^2 + y^2 = h^2 \sin^2 a,$$

$\therefore$ equation to the polar of any fixed point $(\xi, \eta)$ is

$$(x - h)(\xi - h) + y\eta = h^2 \sin^2 a,$$

$$\text{or } h^2 \cos^2 a - h(x + \xi) + x\xi + y\eta = 0.$$

To find the envelope we must express the condition that this equation in $h$ should have equal roots.

$$\therefore (x + \xi)^2 = 4 \cos^2 a (x\xi + y\eta)$$

which is the equation to a parabola.

*This question may also be solved geometrically as follows:—

Let $OL$, $OL'$ be the two given straight lines, and let $OA$ bisect the angle between them, so that $LOA = L'OA = a$. Then if $P$ be any point on $OA$, $P$ is the centre of one of the circles which touch $OL$ and $OL'$, and its radius $= OP \sin a$.

If $B$ be the fixed point, the polar of $B$ is found by joining $PB$, and taking on it a point $R$ such that $PR \,.\, PB = (\text{radius})^2 = OP^2 \sin^2\alpha$, *i.e.* such that $PR \,.\, PB \propto OP^2$, and by drawing a line $RY$ through $R$ perpendicular to $PB$. Thus we see that the problem reduces to the following.

Given $O$ and $B$, two fixed points, and $OA$ a fixed straight line. Then if any point $P$ be taken on $OA$, and $PB$ be joined, and a point $R$ be taken on $PB$ such that $PR \,.\, PB : PO^2$ in a constant ratio, as $P$ moves along $OA$ it is required to find the envelope of a line through $R$ perpendicular to $PB$.

Describe a circle so as to touch $OA$ in $O$ and pass through $B$. Let $BE$ be the diameter through $B$, so that $E$ is a fixed point. Join $PE$ cutting the circle in $Q$. Then $BQE$ is a right angle. Let $RY$, perpendicular to $PB$, meet $PE$ in $Y$, and draw $YS$ parallel to $PB$ meeting $BE$ in $S$.

Then $PR \,.\, PB = PY \,.\, PQ$, and $PO^2 = PE \,.\, PQ$,

$\therefore$ $PY : PE$ in a constant ratio. $\therefore$ the locus of $Y$ is a straight line $YZ$ parallel to $OA$; and $S$ is a fixed point. $\therefore$ the envelope of $RY$ is a parabola of which $S$ is the focus, and $YZ$ the tangent at the vertex.

The property here employed is the useful converse of the proposition that the foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at the vertex.

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## Paper X.

1. $$\frac{x \,.\, r^2 + y \,.\, r + z}{z \,.\, r^2 + y \,.\, r + x} \quad 2$$

$\therefore$ obviously $2z = r + x$, $2y + 1 = r + y$, $2x + 1 = z$,

$\therefore$ $2(x \,.\, r + z) = 2x \,.\, r + 2z = (z - 1)r + r + x = z \,.\, r + x$.

Again $r + x = 2z = 4x + 2$, $\therefore$ $r = 3x + 2$,

$\therefore$ if we give to $x$ in succession the values 1, 2, 3, .... we obtain for $r$ the values 5, 8, 11, ....

$\therefore$ neglecting the scales of notation in which the radix is 1 and 2, we see that the above number can only be found in one scale out of three.

2. The A.M. of 1, 2, 3, .... $3m$ is known to be $>$ the G.M.

$$\therefore \frac{3m(3m+1)}{2} \div 3m > \sqrt[3m]{(3m)!}$$

$\therefore$ it is sufficient to prove that

$$\sqrt[3]{\frac{3m(3m+1)^2}{4}} > \frac{3m+1}{2}, \text{ or that } 3m > \frac{3m+1}{2},$$

or $3m > 1$, which is true if $m > 0$.

3. Todh. *Trig.* Art. 272 gives an expression for $\tan n\theta$ in terms of $\tan\theta$. If we put $\tan\theta = \sqrt{3}$ or $\frac{1}{\sqrt{3}}$, $\theta = \frac{\pi}{3}$ or $\frac{\pi}{6}$. $\therefore$ $n\theta =$ a multiple of $\pi$ or $2\pi$, since $n$ is a multiple of 6. $\therefore$ $\tan n\theta = 0$, and the required results are obtained by equating to zero the numerators of the expressions.

4. Let $BB'$ cut $AA'$ between $A$ and $A'$.

Then $2CBC' = 2\pi - (BCB' + BC'B')$,

$$= 2\pi - \{2(\pi - BAB') + 2(\pi - BA'B')\};$$

$$\therefore CBC' = BAB' + BA'B' - \pi,$$

$$= 2\pi - (ABA' + AB'A') - \pi,$$

$$= \pi - (AOO' + AO'O) = OAO'.$$

If $BB'$ cut $AA'$ produced, $CBC'$ will equal the supplement of $OAO'$.

5. Let the normal at $P$ meet the major axis in $G$. Through $G$ draw a line $LMGL'M'$ perpendicular to the normal, and meeting $SP$ and $PS'$ produced in $M$ and $M'$. Draw $SL$, $S'L'$ perpendicular to $MM'$.

Then $SP : S'P :: SG : S'G :: SL : S'L' :: \frac{SL}{PG} : \frac{S'L'}{PG}$

$$:: \frac{SM}{MP} : \frac{S'M'}{M'P} :: SM : S'M', \text{ since } PM = PM'$$

$$:: SP - PM : PM - PS'.$$

6. To make the highest card project as far as possible from the table, we must move it until it is on the point of falling. Then move forward the one immediately below it until it is on the point of falling, and so on. When the lowest card projects as far as possible from the table, it is evident that the highest card will then project a maximum distance from the table, and that each card will be on the point of falling independently of the rest.

7. Let $S$ be the vertex of the triangle, $S'P$ the base.

If $S'SP = \theta$, $S'PS = 2\theta$, $\cos 2\theta = \frac{\sqrt{5}-1}{4}$, $\cos\theta = \frac{\sqrt{5}+1}{4}$.

Let $SS' = c$, and $e =$ the eccentricity.

Then $2AC = SP + PS' = c + 2c\cos 2\theta = c(1 + 2\cos 2\theta) = 2c\cos\theta$,

$$\therefore \frac{c}{2} = CS = e \,.\, CA = e \,.\, c\cos\theta,$$

$$\therefore e = \frac{1}{2\cos\theta} = \frac{2}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{2},$$

$$\therefore e(e+1) = 1.$$

If $SR$ be semi latus rectum,

$$SR = e(SA + AX) = e(e+1)AX = AX.$$

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## Paper XI.

1. 1000 lbs. = 453·59 kilog. 1000 miles = 1609·27 kilom.

$\therefore$ the question may be stated thus:

If 453·59 kilog. can be carried 1609·27 kilom. for 25·2 francs, how many kilog. can be carried 100 kilom. for 20 francs?

$$\begin{matrix} 100 & : & 1609{\cdot}27 \\ 25{\cdot}2 & : & 20 \end{matrix} :: 453{\cdot}59 : Ans.$$

$$\therefore Ans. = \frac{1609{\cdot}27 \times 453{\cdot}59}{126} = \frac{729948{\cdot}7793}{126}$$

$$= 5793{\cdot}24.$$

2. $\frac{1}{2}(x+y)(a+b-c-d) = cd - ab,$

$$xy(a+b-c-d) = ab(c+d) - cd(a+b),$$

$$\therefore \left(\frac{x-y}{2}\right)^2 (a+b-c-d)^2 = (cd-ab)^2 - (a+b-c-d)\{ab(c+d) - cd(a+b)\}.$$

Now the expression on the right vanishes when $a = c$, or $a = d$, or $b = c$, or $b = d$, and $\therefore$

$= k(a-c)(a-d)(b-c)(b-d)$, where $k$ is some numerical constant. By equating coefficients of any term, as $a^2b^2$ we find $k = 1$.

$$\therefore \left(\frac{x-y}{2}\right)(a+b-c-d) = \sqrt{(a-c)(a-d)(b-c)(b-d)}.$$

3. $\left(1 - \frac{1}{4}\right)^{-\frac{1}{2}} = 1 + \frac{1}{4}\left\{\frac{1}{2} + \frac{1.3}{2.4}\cdot\frac{1}{4} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{4^2} + \ldots\right\};$

$\therefore$ Given expression $= 4\left\{\left(\frac{3}{4}\right)^{-\frac{1}{2}} - 1\right\} = 4\left\{\frac{2}{\sqrt{3}} - 1\right\} = \frac{4}{3}(2-\sqrt{3})\sqrt{3}.$

4. $\sin\left(A + \frac{4\pi}{3}\right) = \sin\left(\pi + A + \frac{\pi}{3}\right) = -\sin\left(A + \frac{\pi}{3}\right)$,

$\sin\left(A + \frac{2\pi}{3}\right) = \sin\left(\pi + A - \frac{\pi}{3}\right) = -\sin\left(A - \frac{\pi}{3}\right)$,

$$\therefore \text{Expression} = \frac{1}{\sin A} - \frac{\sin\left(A + \frac{\pi}{3}\right) + \sin\left(A - \frac{\pi}{3}\right)}{\sin\left(A + \frac{\pi}{3}\right)\sin\left(A - \frac{\pi}{3}\right)},$$

$$= \frac{1}{\sin A} - \frac{2\sin A \cos\frac{\pi}{3}}{\sin^2 A - \sin^2\frac{\pi}{3}};$$

$$= \frac{1}{\sin A} - \frac{4\sin A}{4\sin^2 A - 3} = \frac{3}{3\sin A - 4\sin^3 A}$$

$$= \frac{3}{\sin 3A} = 3\operatorname{cosec} 3A.$$

5. $\tan y = \dfrac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}$;

$$\therefore \frac{1 + \tan y}{1 - \tan y} = \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}},$$

$$\therefore \frac{1 + x^2}{1 - x^2} = \frac{\sec^2 y + 2\tan y}{\sec^2 y - 2\tan y},$$

$$\therefore x^2 = \frac{2\tan y}{\sec^2 y} = 2\sin y \cos y = \sin 2y.$$

6. Tripos 1875. Tuesday morning. No. 6.

7. Let $TP$, $TQ$ be two of the tangents meeting the axes in $t$, $t'$.
Join $ST$ meeting $BC$ in $R$ and draw the ordinate $TN$.
Then the angle $STt = S'Tt$, and $TtS = Tt'R$,

$$\therefore TSt = t'RT = S'TN,$$

$\therefore$ the triangles $STN$, $S'TN$ are similar,

$$\therefore TN^2 = SN \,.\, NS'.$$

$\therefore$ the locus of $T$ is a rectangular hyperbola, having $S$, $S'$ as vertices.
It is evident that $N$ cannot fall between $S$ and $S'$.

## Paper XII.

**1.** Multiply the three equations together

$$\therefore (z+x)(x+y)(y+x) = \pm abc,$$

$$\therefore \text{by division } y+z = \pm \frac{bc}{a},\ z+x = \pm \frac{ca}{b},\ x+y = \pm \frac{ab}{c}.$$

By addition we have $x+y+z = \frac{1}{2}\left\{ \pm \frac{bc}{a} \pm \frac{ca}{b} \pm \frac{ab}{c} \right\}$,

which at once gives us $x$, $y$ and $z$.

**2.** Let $$\frac{x}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b},$$

$$\therefore x \equiv A(x-b) + B(x-a),$$

$$\therefore A+B = 1,\ Ab+Ba = 0,\quad \therefore A = \frac{a}{a-b},\ B = -\frac{b}{a-b},$$

$$\therefore \text{expression} = \frac{1}{a-b}\left\{\frac{b}{b-x} - \frac{a}{a-x}\right\},$$

$$= \frac{1}{a-b}\left\{\left(1-\frac{x}{b}\right)^{-1} - \left(1-\frac{x}{a}\right)^{-1}\right\},$$

$$\therefore \text{coef. of } x^n = \frac{1}{a-b}\left\{\frac{1}{b^n} - \frac{1}{a^n}\right\} = \frac{a^n-b^n}{a-b}\cdot\frac{1}{a^n.b^n}.$$

**3.** Consider the identity

$$(a+b+c)(a^2+b^2+c^2-bc-ca-ab) \equiv a^3+b^3+c^3-3abc.$$

Since this is always true whatever be the values of $a$, $b$, $c$,

$$\text{let } a = e^{\alpha i},\ b = e^{\beta i},\ c = e^{\gamma i}, \text{ where } i = \sqrt{-1},$$

$$\therefore (e^{\alpha i}+e^{\beta i}+e^{\gamma i})\left\{e^{2\alpha i}+e^{2\beta i}+e^{2\gamma i} - e^{(\beta+\gamma)i} - e^{(\gamma+\alpha)i} - e^{(\alpha+\beta)i}\right\}$$

$$= e^{3\alpha i}+e^{3\beta i}+e^{3\gamma i} - 3e^{(\alpha+\beta+\gamma)i}$$

$\therefore$ putting for $e^{\alpha i}$ its value $\cos\alpha + i\sin\alpha$, &c.

$$\left\{\cos\alpha+\cos\beta+\cos\gamma+i(\sin\alpha+\ldots)\right\}$$

$$\left\{\cos 2\alpha+\ldots-\cos(\beta+\gamma)+i(\sin 2\alpha+\ldots)\right\}$$

$$= \cos 3\alpha+\cos 3\beta+\cos 3\gamma - 3\cos(\alpha+\beta+\gamma)+i(\sin 3\alpha+\ldots);$$

$\therefore$ equating real parts we obtain the required equality.

Another relation can at once be written down by equating unreal parts.

4. Let $O$ be the centre of the circum-circle, $P$ the ortho-centre. Let $AP$ produced meet $BC$ in $M$. Through $O$ draw $ON$, $OQ$ perpendicular to $BC$ and $PM$ respectively.

Then $PM = PB \cos C = \dfrac{2c \,.\, \cos B}{2 \sin C} \cos C = 2R \cos B \cos C,$

$$BM = PB \sin C = \frac{2c \,.\, \cos B}{2 \sin C} \sin C = 2R \cos B \sin C,$$

$$ON = R \cos A,\ BN = R \sin A,$$

$$\therefore OP^2 = (PM - ON)^2 + (BM - BN)^2,$$

$$= R^2 \left\{(2 \cos B \cos C - \cos A)^2 + (2 \cos B \sin C - \sin A)^2\right\},$$

$$= R^2 \left\{(\cos \overline{B - C} - 2 \cos A)^2 + \sin^2 (C - B)\right\},$$

$$= R^2 \left\{1 - 4 \cos A \,(\cos \overline{B - C} + \cos \overline{B + C})\right\},$$

$$= R^2 \left\{1 - 8 \cos A \cos B \cos C\right\}.$$

5. Let the circles round $ADE$, $CDF$ meet again in $O$. Join $OD$.

The angle $AOE = ADE = CDF = COF,$

and $FCO = FDO =$ sup. of $ODA = OEA,$

$$\therefore OA : OF :: OE : OC \quad \therefore OA \,.\, OC = OE \,.\, OF.$$

6. Let $P$ be the vertex of the diameter $PO$ which bisects $AB$, and let $PT$, the tangent at $P$, make an angle $\theta$ with the axis.

Then if $x'$, $y'$ be the coordinates of any point $Q$ on the parabola referred to $PO$ and $PT$ as axes,

$$y'^2 = \frac{4a}{\sin^2 \theta}\, x' : \quad \therefore OP = \frac{y'^2}{4a} \,.\, \sin \theta; \quad \therefore \rho = \mu \sin^2 \theta.$$

7. Tripos 1878. Tuesday morning. No. 2.

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## Paper XIII.

1. $$(a + b + c)(a + b - c)(a - b + c)(a - b - c)$$

$$= \left\{(a + b)^2 - c^2\right\} \left\{(a - b)^2 - c^2\right\},$$

$$= \left\{a^2 + b^2 - c^2 + 2ab\right\} \left\{a^2 + b^2 - c^2 - 2ab\right\},$$

$$= (a^2 + b^2 - c^2)^2 - 4a^2b^2,$$

$$= a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2.$$

2. (1) This equation may be written

$$\frac{\left(a^4 + \frac{1}{a^4} + 14\right)^3}{\left(a^4 + \frac{1}{a^4} - 2\right)^2} = \frac{\left(x + \frac{1}{x} + 14\right)^3}{\left(x + \frac{1}{x} - 2\right)^2}.$$

Put $$x + \frac{1}{x} - 2 = 16y,\ a^4 + \frac{1}{a^4} - 2 = 16b,$$

$$\therefore \frac{(y+1)^3}{y^2} = \frac{(b+1)^3}{b^2};$$

$\therefore y = b$ is one solution. The others are found on division to be given by the quadratic $b^2y^2 = b + y$. Then the three values of $x + \frac{1}{x}$, and $\therefore$ the six values of $x$ can be found.

(2) From (1) and (2) by elim. $z$ we get

$$\left.\begin{aligned} &b(a-c)\,x + a(b-c)\,y = a^2b + ab^2 - 2abc \\ \text{So from (1) and (3) } &(a-c)\,x + (b-c)\,y = ab - c^2 \end{aligned}\right\}$$

$$\therefore x = \frac{a(b-c)^2}{(c-a)(a-b)}.$$

The values of $y$ and $z$ can at once be written down by making symmetrical changes of the letters $a$, $b$, $c$.

3. The given expression

$$= \tfrac{1}{4}\left\{\cos 2(A+B) + \cos 2C\right\}\left\{\cos 2(A-B) + \cos 2C\right\}$$
$$+ \tfrac{1}{4}\left\{\cos 2C - \cos 2(A+B)\right\}\left\{\cos 2(A-B) - \cos 2C\right\}$$
$$= \tfrac{1}{2}\left\{\cos 2C \cos 2(A+B) + \cos 2C \cos 2(A-B)\right\},$$
$$= \cos 2C \cos 2A \cos 2B.$$

4. $$(r_2 + r_3)\cdot\sqrt{\frac{r\,r_1}{r_2 r_3}} = S\left(\frac{1}{s-b} + \frac{1}{s-c}\right)\cdot\sqrt{\frac{(s-b)(s-c)}{s\,.\,s-a}},$$
$$= S\,.\,\frac{2s-(b+c)}{(s-b)(s-c)}\sqrt{\frac{(s-b)(s-c)}{s\,.\,(s-a)}},$$
$$= 2s - (b+c) = a.$$

5. The angle $EAC = \frac{1}{2}ABC = ABD = ACD$,

$\therefore EA$ is parallel to $CD$.

The angle $ECA = \frac{1}{2} ACB = CBD = CAD,$

$\therefore EC$ is parallel to $AD$.

6. Let $QNR$ be the chord cutting the major axis in $N$. Let $A'Q$ meet the tangent at $A$ in $L$. Then the angle $QAL = QA'A$;

$\therefore QAN + QA'N = QAN + QAL =$ one right angle.

$\therefore QAR + QA'R =$ two right angles.

7. Let $G$ be the centre. Then it is evident that for any given weight to have the greatest effect in upsetting the table, it must be placed at one of the corners, $A$ suppose. Let $E$, $F$ be the middle points of the adjacent sides. Let $W$ be the weight of the table, and let $P$ be the greatest weight which can be placed at $A$ without destroying equilibrium. Let $AG$ and $EF$ intersect in $H$. Taking moments round $H$, we have

$$P \,.\, AH = W \,.\, HG, \ \therefore P = W.$$

$\therefore$ no weight less than the table when placed upon it can upset it.

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## Paper XIV.

1. $(ax - by)^2 + (ax - by)(ay + bx + by) + (ay + bx + by)^2$

$= a^2x^2 - 2abxy + b^2y^2 + a^2xy + abx^2 + abxy - aby^2 - b^2xy - b^2y^2 + a^2y^2$
$+ b^2c^2 + b^2y^2 + 2abxy + 2b^2xy + 2aby^2,$

$= a^2x^2 + a^2xy + a^2y^2 + abx^2 + abxy + aby^2 + b^2x^2 + b^2xy + b^2y^2,$

$= (x^2 + xy + y^2)(a^2 + ab + b^2).$

*We also give the following instructive method of proving the above.

Let $1, w, w^2$ be the three cube roots of unity, so that

$$w^2 + w + 1 = 0 \ \therefore w^2 = -w - 1$$

and $$w^3 = 1 \ \therefore w^4 = w = -1 - w^2.$$

Then $$x^2 + xy + y^2 = (x - wy)(x - w^2y),$$

$$\therefore (x^2 + xy + y^2)(a^2 + ab + b^2)$$

$= \{(x - wy)(a - wb)\} \{(x - w^2y)(a - w^2b)\},$

$= \{ax + w^2by - w(ay + bx)\} \{ax + w^4by - w^2(ay + bx)\},$

$= \{(ax - by) - w(ay + bx + by)\} \{ax - by - w^2(ay + bx + by)\},$

$= (ax - by)^2 - (ax - by)(ay + bx + by)(w + w^2) + w^3(ay + bx + by)^2,$

$= (ax - by)^2 + (ax - by)(ay + bx + by) + (ay + bx + by)^2.$

Thus we see that the product of these two factors is of the form

$$X_1^2 + X_1Y_1 + Y_1^2.$$

$\therefore$ let $(x_1^2 + x_1y_1 + y_1^2)(x_2^2 + x_2y_2 + y_2^2) = X_1^2 + X_1Y_1 + Y_1^2.$

Introduce a new factor $x_3^2 + x_3y_3 + y_3^2$. Then the product of the three factors

$$= (X_1^2 + X_1Y_1 + Y_1^2)(x_3^2 + x_3y_3 + y_3^2),$$

$$= X_2^2 + X_2Y_2 + Y_2^2 \text{ suppose.}$$

Hence we see that the product of any number of factors of the form $x^2 + xy + y^2$ can be put in the form $X^2 + XY + Y^2$.

2. $$\frac{b - lz + nx}{m'} = \frac{c - mx + ly}{n'},$$

$$\therefore x(mm' + nn') - (m'c - n'b) = l(m'y + n'z),$$

$$\therefore x(ll' + mm' + nn') - (m'c - n'b) = l(l'x + m'y + n'z),$$

$$\therefore \frac{l'x + m'y + n'z}{ll' + mm' + nn'} = \frac{x - \dfrac{m'c - n'b}{ll' + mm' + nn'}}{l}.$$

Similarly it may be shewn that $\dfrac{l'x + m'y + n'z}{ll' + mm' + nn'}$ is equal to each of the other expressions.

3. $\dfrac{A}{2} + \dfrac{B}{2} + \dfrac{C}{2} + \dfrac{D}{2} = \pi;$

$$\therefore \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left\{\pi - \frac{C}{2} - \frac{D}{2}\right\} = -\tan\left(\frac{C}{2} + \frac{D}{2}\right);$$

$$\therefore 0 = \tan\left(\frac{A}{2} + \frac{B}{2}\right) + \tan\left(\frac{C}{2} + \frac{D}{2}\right),$$

$$= \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} + \frac{\tan\frac{C}{2} + \tan\frac{D}{2}}{1 - \tan\frac{C}{2}\tan\frac{D}{2}},$$

$$= \left(\tan\frac{A}{2} + \tan\frac{B}{2}\right)\left(1 - \tan\frac{C}{2}\tan\frac{D}{2}\right) + \left(\tan\frac{C}{2} + \tan\frac{D}{2}\right)\left(1 - \tan\frac{A}{2}\tan\frac{B}{2}\right);$$

$$\therefore \left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\tan\frac{C}{2}\tan\frac{D}{2}+\left(\tan\frac{C}{2}+\tan\frac{D}{2}\right)\tan\frac{A}{2}\tan\frac{B}{2}$$

$$=\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}+\tan\frac{D}{2}.$$

Now $\tan\frac{A}{2}=\frac{r}{a}$, &c.

$$\therefore \left(\frac{r}{a}+\frac{r}{b}\right)\cdot\frac{r}{c}\cdot\frac{r}{d}+\left(\frac{r}{c}+\frac{r}{d}\right)\frac{r}{a}\cdot\frac{r}{b}=\frac{r}{a}+\frac{r}{b}+\frac{r}{c}+\frac{r}{a},$$

$$\therefore r^2=\frac{bcd+cda+dab+abc}{a+b+c+d}.$$

4.
$$\sin 2\theta-\sin\theta=2\sin\theta\cos\theta,$$
$$\sin 4\theta-\sin 2\theta=2\sin\theta\cos 3\theta,$$
$$\sin 6\theta-\sin 4\theta=2\sin\theta\cos 5\theta,$$
$$\ldots\ldots$$
$$\sin 2^n\theta-\sin(2^n-2)\theta=2\sin\theta\cos(2^n-1)\theta,$$

$\therefore$ by addition we find

$$\cos\theta+\cos 3\theta+\cos 5\theta+\ldots.+\cos(2^n-1)\theta=\frac{\sin 2^n\theta}{\sin\theta},$$

$=2^{n-1}\cos\theta\cos 2\theta\cos 4\theta\ldots.\cos 2^{n-1}\theta$. Todh. *Trig.* Art. 129.

5. Let one force act at $A$ and the other two at $B$, and let each force be denoted by $P$. Then the resultant of the two forces at $B$ must be a force $P$ parallel to the force at $A$. $\therefore$ the forces at $B$ must each make an angle 60° with the resultant, and $\therefore$ must make an angle 120° with each other.

6. We will first prove the following useful proposition.

'The perpendicular from the focus on any tangent and the diameter through the point of contact intersect on the directrix.'

Draw $SY$ perpendicular to the tangent at $P$ and produce $SY$ to meet the directrix in $D$. Draw the tangents $DQ$, $DQ'$. Then since the directrix is the polar of the focus, $QQ'$ passes through $S$, and is perpendicular to $SD$. Bes. *Con.* p. 10.

$\therefore$ $QQ'$ is parallel to the tangent at $P$. $\therefore$ the diameter through $P$ bisects $QQ'$ and passes through $D$, the point of intersection of tangents at $Q$ and $Q'$.

From this the question in the text follows at once.

7. Draw $CE$ perpendicular to $AB$.

Then $CD^2 = DE^2 - EC^2 = (DA - AE)^2 + (AC^2 - AE^2)$

$= DA^2 + AE^2 - 2DA \,.\, AE + AB^2 - AE^2$

$= DA^2 - AD \,.\, 2AE + AB^2$

$= 9AB^2 - 3AB^2 + AB^2 = 7AB^2.$

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## Paper XV.

1. Let $P$ denote the population at the beginning of the 1st year, and let $P_1$, $P_2$, $P_3$ denote it at the end of the 1st, 2nd, 3rd .... years.

Then $P_1 = P + \left(\frac{1}{60} - \frac{1}{90}\right)P = P + \frac{P}{180} = \frac{181}{180}P.$

So $P_2 = \frac{181}{180}P_1 = \left(\frac{181}{180}\right)^2 P$; and so on.

$\therefore$ if $x$ denote the required number of years

$$\left(\frac{181}{180}\right)^x P = 2P,$$

$$\therefore x\,(\log 181 - \log 180) = \log 2.$$

$$\therefore x = \frac{\cdot 301030}{\cdot 002407} = 125\cdot 06 \text{ years nearly.}$$

2. Employing the method given in the notes 1, (1) we have

$$\frac{x^2}{bc' - b'c} = \frac{xy}{ca' - c'a} = \frac{y^2}{ab' - a'b'}$$

$$\therefore (bc' - b'c)(ab' - a'b) = (ca' - c'a)^2.$$

3. Take logarithms of both sides,

$$\therefore (x + 2)(1 - \log 3) = 2(2x - 1)\log 3,$$

$$\therefore x = \frac{2}{5\log 3 - 1} = \frac{1}{\cdot 69280325} = 1\cdot 428\ldots$$

4. The angle $BOP = \frac{A}{2} + \frac{B}{2} = 90^\circ - \frac{C}{2}$,

$$BP = PC = \frac{a}{2}\sec\frac{A}{2},$$

radius of circle round $BOP$

$$= \tfrac{1}{2} BP \operatorname{cosec} BOP = \frac{a}{4} \frac{1}{\cos\frac{A}{2}\cos\frac{C}{2}} = \frac{a\sqrt{acb^2}}{4\sqrt{s^2(s-a)(s-c)}}$$

$$= \frac{ab\sqrt{ca(s-b)}}{4r \cdot s^{\frac{3}{2}}}.$$

$\therefore$ product of radii of the first three circles is

$$= \frac{a^2b^2c^2 \cdot abc\sqrt{(s-a)(s-b)(s-c)}}{4^3 \cdot r^3 \cdot s^{\frac{9}{2}}}$$

$$= \frac{a^3b^3c^3 \cdot S}{4^3 \cdot r^3 \cdot s^5} = \frac{a^3b^3c^3 \cdot r}{4^3 \cdot r^3 \cdot \left(\frac{a+b+c}{2}\right)^4} = \frac{a^3b^3c^3}{4r^2(a+b+c)^4}.$$

Similarly it may be shewn that the product of the second three radii is equal to the same expression.

5. Through $B$, $C'$ draw $BE$, $C'E$ parallel respectively to $CC'$, $CB$. Then $BB'E$ shall fulfil the given conditions. For in the triangles $B'EC$, $ADD'$, $AD$ is equal and parallel to $EC'$, for each is equal and parallel to $BC$; and $AD'$ is equal and parallel to $B'C'$, $\therefore$ $B'E$ is equal and parallel to $DD'$.

6. Let $TP$, $TP'$ be the tangents, $Q$ any point in $ST$. Draw $SY$, $QZ$ perpendicular to $TP$, and $SY'$, $QZ'$ perpendicular to $TP'$.

Then $YY'$ is the tangent at the vertex.

Also $$SY : QZ :: ST : QT$$
$$:: SY' : QZ',$$

and the angle $YSY' = ZQZ'$, $\therefore$ $ZZ'$ is parallel to $YY'$.

7. Since the conic has double contact with the circle, it is of the form

$$x^2 + y^2 - a^2 + (lx + my - n)^2 = 0.$$

Since it passes through the origin, the absolute term $= 0$.

$$\therefore n^2 = a^2.$$

Since the axis of $x$ is a tangent at the origin, $\therefore$ by putting $y = 0$ the expression $x^2 - a^2 + (lx - a)^2$ must reduce to $x^2 = 0$.

$\therefore l = 0$, and the equation becomes

$$x^2 + (1 + m^2) y^2 - 2amy = 0,$$

or, writing $\frac{a}{c}$ for $m$,

$$c^2x^2 + (a^2 + c^2) y^2 - 2a^2cy = 0.$$

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## Paper XVI.

1. Let $d$ denote the distance from Cambridge to Oxford. Then $\frac{d}{u}$, $\frac{d}{v}$ are the rates of $A$ and $B$. When they meet $A$ has walked $u - \alpha$ hours, and $B$ $v - \beta$ hours, $\therefore u - \alpha = v - \beta$;

$\therefore$ when they meet, $A$ has walked $\frac{d}{u}(u - \alpha)$ miles, and $B$ $\frac{d}{v}(v - \beta)$.

$$\therefore \frac{d}{u}(u - \alpha) + \frac{d}{v}(v - \beta) = d,$$

$$\therefore v(u - \alpha) + u(v - \beta) = uv,$$

$$\therefore (u + v)(u - \alpha) = uv, \quad (1)$$

$$(u + v)(v - \beta) = uv, \quad (2) \text{ since } u - \alpha = v - \beta.$$

From (1) $u^2 = \alpha(u + v)$; From (2) $v^2 = \beta(u + v)$;

$$\therefore u^2 : v^2 :: \alpha : \beta.$$

2. Clearing the second equation of fractions we get

$$a(x^2 + y^2) + xy(x + y) = a^3 + a^2(x + y) + axy,$$

$$\therefore 2a^3 + xy(x + y) = a^3 + a^2(x + y) + axy,$$

$$\therefore (x + y - a)(xy - a^2) = 0.$$

$\therefore$ either $x + y = a$; (1) or $xy = a^2$. (2).

From (1) $y = a - x$, $\therefore x^2 + (a - x)^2 = 2a^2$, $\therefore x = \frac{a}{2}(1 \pm \sqrt{3})$.

$$\therefore y = a - x = \frac{a}{2}(1 \mp \sqrt{3}).$$

From (2) $(x - y)^2 = 0$, $\therefore x = y = \pm a$. The negative sign is seen by the 2nd equation to be inadmissible.

3. Let $B$ denote the foot of the tree, $BA$ its height, $C$ and $D$ the first and second positions of the observer. Then $BCA = 51°$ $BDA = 46°$, $DAC = 5°$, $BAC = 39°$, $DC = 30$ feet

$$AC = 30 \cdot \frac{\sin 46°}{\sin 5°},$$

$$\therefore \log AC = \log 30 + L \sin 46° - L \sin 5°$$
$$= 1{\cdot}417712 + 9{\cdot}856934 - 8{\cdot}940296$$
$$= 2{\cdot}334350.$$

$$BC = AC \cdot \sin 39°,$$

$$\therefore \log BC = \log AC + L \sin 39° - 10$$
$$= 2{\cdot}334350 + 9{\cdot}798872 - 10$$
$$= 2{\cdot}133222,$$

$$\log 135{\cdot}90 = 2{\cdot}133219,$$

$$\log 135{\cdot}91 = 2{\cdot}133251,$$

$$\therefore BC = 135{\cdot}90093 \ldots$$

NOTE.—The question is given as it was originally stated. The value of log 3 is incorrect, and the value of $AB$ would be found to be about 190 feet, which is a good height for a tree.

4. $$f(2\theta) = (1 - \tan^2\theta)\, f(\theta) = \frac{2 \tan\theta}{\tan 2\theta} \cdot f(\theta),$$

$$\therefore \frac{f(2\theta)}{2\theta \cot 2\theta} = \frac{f(\theta)}{\theta \cot \theta} = m,$$

$$\therefore f(\theta) = m \cdot \theta \cot \theta.$$

5. Let $A$ and $B$ be the given points, $CD$ the length of the given straight line. In the given circle place a straight line equal to $CD$. The angle in the segment which this straight line cuts off is known. On $AB$ describe a circle containing an angle equal to this angle, and let the two circles intersect in $G$ and $H$. Let $AG$, $GB$ cut the circle in $E$, $F$, and let $AH$, $HB$ cut the circle in $K$, $L$. Then the triangles $EFG$, $HKL$ satisfy the given conditions. For the angles $EGF$, $KHL$ are each equal to the angle in a segment cut off by a straight line equal to $CD$. $\therefore$ $EF$ and $KL$ are each equal to $CD$.

6. Let $ABCD$ be the tetrahedron. At $A$, $B$, $C$, $D$ place weights respectively proportional to the lengths of the tangents from these points to the sphere. Let the sphere touch the edges $AD$, $BC$ in $P$, $Q$. Then $P$ is the C. of G. of the weights at $A$, $D$; $Q$ the C. of G. of those

at $B$, $C$. $\therefore$ the C. of G. of the four weights lies in $PQ$. Similarly the C. of G. of the system may be shewn to lie in each of the other two lines joining the points of contact of the sphere with the opposite edges $AB$, $CD$ and $AC$, $BD$. $\therefore$ these three lines are collinear.

7. Let $xy = k^2$ be the equation to the hyperbola referred to the asymptotes as axes. The equation to the ellipse will be

$$xy = \lambda^2 \left(\frac{x}{a} + \frac{y}{b} - 1\right)^2, \text{ where } \frac{x}{a} + \frac{y}{b} = 1,$$

is the equation to the chord of contact. Then the equations to $PP'$ and $QQ'$ are evidently

$$\frac{x}{a} + \frac{y}{b} - 1 = \pm \frac{k}{\lambda},$$

and $\therefore$ these two lines are parallel to the chord of contact.

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## Paper XVII.

1. By ordinary division we find the quotient is

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9.$$

2. We may consider $a$, $b$, $c$ as roots of the equation

$$\frac{x}{k + a} + \frac{y}{k + \beta} + \frac{z}{k + \gamma} = 1,$$

which is of the 3rd degree in $k$. Assume $k + a = \lambda$;

$$\therefore k + \beta = \lambda + \beta - a, \text{ \&c. } \therefore a + a,\ a + b,\ a + c$$

are the roots of

$$\frac{x}{\lambda} + \frac{y}{\lambda + \beta - a} + \frac{z}{\lambda + \gamma - a} = 1.$$

$\therefore$ by multiplication, we have

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0,$$

where $$A_3 = -x(\beta - a)(\gamma - a).$$

Also, since the product of the roots is equal to the last term with its sign changed,

$$x(a - \beta)(a - \gamma) = \lambda_1 . \lambda_2 . \lambda_3 = (a + a)(b + a)(c + a).$$

3. Since $\cos\alpha + \cos\beta + \cos\gamma = 0,$

and $\sin\alpha + \sin\beta + \sin\gamma = 0,$

$$\therefore e^{\alpha i} + e^{\beta i} + e^{\gamma i} = 0.$$

Now whatever be the values of $a, b, c$, if $a + b + c = 0$,

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$$
$$= 0.$$

$\therefore$ writing $e^{\alpha i}$ for $a$, &c.

$$e^{3\alpha i} + e^{3\beta i} + e^{3\gamma i} = 3e^{(\alpha+\beta+\gamma)i}$$

$\therefore$ equating real and unreal parts, we obtain the required relations.

4. Let $DEF$ be the triangle, and let $DP$ produced meet the circle in $C$.

Then $$DP = \frac{r}{\sin\frac{D}{2}};$$

and the angle $CPE = PDE + PED = CDB + PEF = CEP$,

$$\therefore CP = CE = 2R\sin\frac{D}{2},$$

$$\therefore PA \,.\, PB = PD \,.\, PC = 2Rr.$$

5. Let $ABCD$ be the quadrilateral; $E, F, G, H$ the middle points of $AB, BC, CD, DA$. Suppose four equal weights to be placed, one at each corner. Then the C. of G. of those at $A$ and $B$ is at $E$, and the C. of G. of those at $C$ and $D$ is at $G$. $\therefore$ the C. of G. of the system is at the middle point of $EG$. Similarly it may be shewn to be at the middle point of $FH$. $\therefore$ $EG$ and $FH$ bisect each other.

This may also be proved by geometry, by shewing that $EFGH$ is a parallelogram, for two sides are parallel to $BD$, and the other two are parallel to $AC$, and $\therefore$ $EG$ and $FH$, which are its diagonals, bisect each other.

Again, let $EFGH$ be the given parallelogram. Through $E$ and $H$ draw *any* straight lines $AEB$, $AHD$, and take $EB = EA$ and $DH = HA$. Join $BF$, $DG$ and produce them to meet in $C$. Then $BD = 2HE = 2GF$, since $EFGH$ is a parallelogram. And $GF$ is parallel to $BD$,

$$\therefore CF : CB :: FG : BD :: 1 : 2.$$

$\therefore$ $F$ is the middle point of $BD$. Similarly $G$ is the middle point of $CD$. Since $AB$, $AD$ are *any* straight lines through $E$ and $H$, an infinite number of quadrilaterals can be described having their sides bisected at the points $E, F, G, H$.

Again, considering any one of the quadrilaterals $ABCD$ the triangle $AHE$ is $\frac{1}{4}$ $ADB$, and $GFC$ is $\frac{1}{4}$ $CBD$. $\therefore$ $AHE$ and $GFC$ are together $\frac{1}{4}$ the quadrilateral. Similarly $BEF$ and $DGH$ are together $\frac{1}{4}$ the quadrilateral. $\therefore$ the four triangles are together equal to $\frac{1}{2}$ $ABCD$, and $\therefore$ the quadrilateral $ABCD$ is double the remainder, $EFGH$.

6. Let $O$ be the vertex of the cone, $PCP'$ any diameter of the section $APA'$. Join $OP$, $OP'$. Let the focal sphere which is nearest $O$ touch $OP$ in $R$, and let the other focal sphere touch $OP'$ in $R'$.

Then
$$\begin{aligned} OP + OP' &= OR + RP + OR' - R'P' \\ &= OR + SP + OR' - P'S' \\ &= OR + OR', \text{ since } SPS'P' \text{ is a parallelogram,} \\ &= \text{const.} \end{aligned}$$

7. Tripos 1875. Tuesday morning. No. 1.

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## Paper XVIII.

1. Tripos 1875. Monday afternoon. No. 3.

2. Tripos 1875. Monday afternoon. No. 3.

3. Tripos 1875. Monday afternoon. No. 8.

4. $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$; $\therefore$ $\frac{A}{2} + C = \frac{\pi}{2} - \frac{B - C}{2}$;

$\therefore$ $\tan\left(\frac{A}{2} + C\right) = \cot\frac{B - C}{2} = \frac{b + c}{b - c}\tan\frac{A}{2}$. Todh. *Trig.* Art. 229.

$\therefore$ $\frac{(b - c)(b + c - a)}{b + c}\tan\left(\frac{A}{2} + C\right) = (s - a)\tan\frac{A}{2}$ = diameter of inscr. circle.

Similarly it may be shewn that each of the other expressions is equal to $2r$.

5. Let $ABCD$ be the given quadrilateral, $O$ the intersection of its diagonals, and $A'$, $B'$, $C'$, $D'$ the feet of the perpendiculars on the diagonals from $A$, $B$, $C$, $D$. Then because a circle will go round $BCB'C'$, $\therefore$ the angle $OBC = OB'C'$. Similarly $OBA = OB'A'$. $\therefore$ the angle $ABC = A'B'C'$. In the same manner it may be shewn that the

other corresponding angles are equal. Again, from similar triangles $OBC$, $OB'C'$,

$$BC : B'C' :: OB : OB'$$
$$:: AB : A'B' \text{ from } AOB, A'OB'.$$

$\therefore$ the sides about the equal angles $ABC$, $A'B'C'$ are proportionals. The same property may be shewn to hold for the sides about the other equal angles. $\therefore$ the quadrilaterals are similar.

6. Let $C$ be the centre of the circle on $AB$ as diameter, $KK'$ the directrix touching the concentric circle in $X$. Draw $AK$, $BK'$ perpendiculars from $A$, $B$. With centre $A$, radius $AK$, describe a circle. With centre $B$, and radius $BK'$, describe a circle, and let these two intersect in $S$, $S'$. Then $S$, $S'$ are the two positions of the focus corresponding to the position $KK'$.

$$SA + SB = AK + BK' = 2CX = \text{constant.}$$

$\therefore$ the locus of $S$ is an ellipse having $A$, $B$ for foci.

7. Tripos 1875. Tuesday morning. No. 2.

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## Paper XIX.

1. $(a+b)^3(a^5+b^5) = (a^3+3a^2b+3ab^2+b^3)(a^5+b^5)$
$= a^8+3a^7b+8a^6b^2+a^5b^3+a^3b^5+3a^2b^6+3ab^7+b^8 = A,$
$5ab(a+b)^2(a^4+b^4) = 5ab(a^2+2ab+b^2)(a^4+b^4)$
$= 5a^7b+10a^5b^2+5a^5b^3+5a^3b^5+10a^2b^6+5ab^7 = B,$
$15a^2b^2(a+b)(a^3+b^3) = 15a^5b^2+15a^5b^3+15a^3b^5+15a^2b^6 = C,$
$\therefore A+B+C+35a^3b^3(a^2+b^2)+70a^4b^4$
$= a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$
$= (a+b)^8.$

2. Tripos 1875. Monday afternoon. No. 5.

3. $a + b + c = 0$; $\therefore (a+b)^5 = -c^5$;

$$\therefore -\frac{a^5+b^5+c^5}{5} = \frac{(a+b)^5 - a^5 - b^5}{5}$$
$$= ab\{a^3 + 2a^2b + 2ab^2 + b^3\}.$$

$$\text{So} - \frac{a^3 + b^3 + c^3}{3} = \frac{(a+b)^3 - a^3 - b^3}{3}$$

$$= ab(a+b),$$

$$\text{and} \quad (a+b)^2 = c^2,$$

$$\therefore \frac{a^2 + b^2 + c^2}{2} = \frac{(a+b)^2 + a^2 + b^2}{2}$$

$$= a^2 + ab + b^2 = (a+b)^2 - ab,$$

$$\therefore \frac{a^5 + b^5 + c^5}{5} = -ab\{(a+b)^3 - ab(a+b)\},$$

$$= \frac{a^3 + b^3 + c^3}{3} \cdot \frac{a^2 + b^2 + c^2}{2}.$$

From the given conditions, we have $\epsilon^{\alpha i} + \epsilon^{\beta i} + \epsilon^{\beta i} = 0$.

$\therefore$ in the above result writing $\epsilon^{\alpha i}$ for $a$, &c. we have

$$\frac{\epsilon^{5\alpha i} + \epsilon^{5\beta i} + \epsilon^{5\gamma i}}{5} = \frac{\epsilon^{3\alpha i} + \epsilon^{3\beta i} + \epsilon^{3\gamma i}}{3} \cdot \frac{\epsilon^{2\alpha i} + \epsilon^{2\beta i} + \epsilon^{2\gamma i}}{2}$$

$\therefore$ putting for $\epsilon^{5\alpha i}$ its value $\cos 5\alpha + i \sin 5\alpha$, &c., and equating real parts, we obtain the required relation.

If we equate unreal parts, we have

$$\frac{\sin 5\alpha + \sin 5\beta + \sin 5\gamma}{5}$$

$$= \frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{3} \cdot \frac{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}{2}$$

$$+ \frac{\sin 3\alpha + \sin 3\beta + \sin 3\gamma}{3} \cdot \frac{\cos 2\alpha + \cos 2\beta + \cos 2\gamma}{2}.$$

4. Tripos 1875. Monday afternoon. No. 9.

5. Tripos 1875. Monday morning. No. 2.

6. Let $D$ be the middle point of $BC$, $G$ the C. of G. of the triangle.

Then $P = \frac{1}{3}W$. $\therefore$ by symmetry the force will just be able to lift a corner of the triangle if applied at $B$ or $C$.

7. Produce $SY$ to meet $S'P$ in $K$. Then $PRYK$ is a parallelogram,

$$\therefore PR = KY = SY.$$

## Paper XX.

1. $\dfrac{3-\sqrt{5}}{2} = \dfrac{1}{3} + \dfrac{1}{3.7} + \dfrac{1}{3.7.47} + \dfrac{1}{3.7.47.2207} + \ldots$

$= \cdot\dot{3} + \cdot 04761904762 + \cdot 00101317123 + \cdot 00000045907 + \ldots$

$= \cdot 38196601125.$

$\therefore \sqrt{5} = 3 - \cdot 76393202250,$

$= 2 \cdot 2360679775.$

2. $3^{2n+2} - 8n - 9 = 9^{n+1} - 1 - 8n - 8,$

$= (9-1)\{9^n + 9^{n-1} + \ldots + 9 + 1 - n - 1\},$

$= 8\{9^n - 1 + 9^{n-1} - 1 + \ldots + 9 - 1\}.$

The quantity in the brackets is evidently a multiple of $9 - 1$;

$\therefore$ the given expression is a multiple of 64.

Another method of solving this class of problems is as follows:—

By trial we find that the expression is a multiple of 64 when $n = 1$ and when $n = 2$. Suppose that it is a multiple of 64 when $n = p$. Then writing $p + 1$ for $n$, and subtracting the value of the expression when $n = p$, we have

$$9^{p+1} - 8(p+1) - 9 - \{9^p - 8p - 9\},$$

$$\text{which} = 9^p(9-1) - 8,$$

$$= 8(9^p - 1),$$

and $9 - 1$ will evidently divide $9^p - 1$. Thus we see that if the expression is a multiple of 64 when $n = p$, it is also a multiple of 64 when $n = p + 1$. Now we know by trial that this is the case when $n = 1$, and when $n = 2$; $\therefore$ it is the case universally.

3. Tripos 1878. Monday afternoon. No. 8.

4. Tripos 1878. Monday afternoon. No. 9

5. Tripos 1878. Tuesday morning. No. 3.

6. Tripos 1878. Monday morning. No. 1.

7. Tripos 1875. Monday morning. No. 10.

## Paper XXI.

1. Tripos 1878. Monday afternoon. No. 1.

2. Tripos 1878. Monday afternoon. No. 2.

3. Tripos 1875. Monday afternoon. No. 10.

4. Tripos 1878. Monday afternoon. No. 8.

5. Tripos 1878. Tuesday morning. No. 4.

6. Tripos 1878. Monday morning. No. 4.

7. Let $PQR$ be the triangle formed by the tangents. Draw $PB$, $QC$ parallel to the axis, and make at $P$ and $Q$ the angles $RPS$, $RQS$ equal to the supplement of $BPQ$. Then $S$ is the focus. Draw $SD$ and $SE$ perpendicular to $PQ$, $PR$. Then $DE$ is the tangent at the vertex, and $SA$, perpendicular to $DE$, is the axis, and the parabola can be described.

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## Paper XXII.

1. For $x$ write $\tan A$, &c.

$$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C;$$

$$\therefore A + B + C = n \,.\, 180°;$$

$$\therefore 2A + 2B + 2C = 2n \,.\, 180°;$$

$$\therefore \tan 2A + \tan 2B + \tan 2C = \tan 2A \,.\, \tan 2B \,.\, \tan 2C;$$

$$\text{and } \tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{2x}{1 - x^2}, \text{ &c.}$$

2. Tripos 1878. Monday afternoon. No. 3.

3. $$\frac{a^7 + b^7 + c^7}{7} = \frac{a^7 + b^7 - (a+b)^7}{7},$$

$$= -ab\left\{a^5 + 3a^4b + 5a^3b^2 + 5a^2b^3 + 3ab^4 + b^5\right\},$$

$$\frac{a^5 + b^5 + c^5}{5} = -ab\left\{a^3 + 2a^2b + 2ab^2 + b^3\right\},$$

$$\frac{a^2 + b^2 + c^2}{2} = a^2 + ab + b^2,$$

$\therefore$ by multiplication we find

$$\frac{a^7 + b^7 + c^7}{7} = \frac{a^5 + b^5 + c^5}{5} \cdot \frac{a^2 + b^2 + c^2}{2}.$$

Then proceeding exactly as in XIX. 3, we obtain the required relation.

4. Tripos 1875. Monday afternoon. No. 11.

5. Let $ABCD$ be the tetrahedron. Let the resultant of the forces along $DB$ and $DC$ cut $BC$ in $E$. Let them be replaced by this resultant. Then the resultant of this force and the force along $DA$ will cut the plane $ABC$ in a point in $AE$, and is $\therefore$ neither wholly in, nor parallel to this plane. But the action of the forces along $AB$, $BC$, $CA$ lies wholly in the plane $ABC$. $\therefore$ the system cannot be in equilibrium.

6. Tripos 1878. Monday morning. No. 3.

7. Tripos 1878. Monday afternoon. No. 9.

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## Paper XXIII.

1. Tripos 1878. Wednesday morning. No. 3.

2. Tripos 1878. Monday afternoon. No. 3.

3. Let $AB$ be the rod, $A$ the hinge, $C$ the point to which the other end of the rod is fastened by the string, so that $AC = AB$. Draw $AF$ perpendicular to $BC$, and let the vertical through the middle point of $AB$ intersect $BC$ in $G$. Then the action of the hinge at $A$ must pass through $G$. Let $T$, $W$, $R$ denote the tension of the string, the weight of the rod, and the reaction of the hinge. Then taking moments about $A$, $2a$ being the length of the rod, and $\theta$ the angle which it makes with the horizon,

$$W \cdot \frac{a}{2}\cos\theta = T \cdot a \sin\frac{\theta}{2} \text{ and } T = W; \therefore \frac{1}{2}\cos\theta = \sin\frac{\theta}{2};$$

$$\therefore \sin^2\frac{\theta}{2} + \sin\frac{\theta}{2} - \frac{1}{2} = 0; \therefore \sin\frac{\theta}{2} = \frac{-1 \pm \sqrt{3}}{2}.$$

Taking the positive sign we have

$$\cos^2\frac{\theta}{2} = 1 - \sin^2\frac{\theta}{2} = 1 - \frac{4 - 2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} = \sqrt{\frac{3}{4}};$$

$$\therefore \cos\frac{\theta}{2} = \left(\frac{3}{4}\right)^{\frac{1}{4}}; \quad \therefore \theta = 2\cos^{-1}\left(\frac{3}{4}\right)^{\frac{1}{4}}.$$

4. (1) $$\sin 6A + \sin 6B + \sin 6C$$
$$= 2 \sin (3A + 3B) \cos (3A - 3B) + 2 \sin 3C \cos 3C$$
$$= 2 \sin 3C \{\cos (3A - 3B) - \cos (3A + 3B)\}$$
$$= 2 \sin 3C \sin 3A \sin 3B.$$

(2) $$\tan \left(\frac{\pi}{4} - \frac{A}{4}\right) = \frac{1 - \tan \frac{\pi}{4}}{1 + \tan \frac{\pi}{4}}, \text{ \&c.}$$

$\therefore$ the given expression

$$= \frac{\sin \left(\frac{\pi}{4} - \frac{A}{4}\right) \sin \left(\frac{\pi}{4} - \frac{B}{4}\right) \sin \left(\frac{\pi}{4} - \frac{C}{4}\right)}{\cos \left(\frac{\pi}{4} - \frac{A}{4}\right) \cos \left(\frac{\pi}{4} - \frac{B}{4}\right) \cos \left(\frac{\pi}{4} - \frac{C}{4}\right)} = \frac{N}{D}.$$

Now $$\frac{B + C}{2} = \frac{\pi}{2} - \frac{A}{2}, \quad \therefore \frac{B + C}{4} = \frac{\pi}{4} - \frac{A}{4}, \text{ \&c.}$$

$$\therefore N = \sin \frac{B + C}{4} \sin \frac{C + A}{4} \sin \frac{A + B}{4}$$
$$= \frac{1}{2} \sin \frac{B + C}{4} \left(\cos \frac{B - C}{4} - \cos \frac{2A + B + C}{4}\right)$$
$$= \frac{1}{4} \left(\sin \frac{B}{2} + \sin \frac{C}{2}\right) - \frac{1}{4} \left(\sin \frac{2(A + B + C)}{4} - \sin \frac{A}{2}\right)$$
$$= \frac{1}{4} \left(\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} - 1\right),$$

$$D = \cos \frac{B + C}{4} \cos \frac{C + A}{4} \cos \frac{A + B}{4}$$
$$= \frac{1}{2} \cos \frac{B + C}{4} \left(\cos \frac{B - C}{4} + \cos \frac{2A + B + C}{4}\right)$$
$$= \frac{1}{4} \left(\cos \frac{B}{2} + \cos \frac{C}{2}\right) + \frac{1}{4} \left(\cos \frac{2(A + B + C)}{4} + \cos \frac{A}{2}\right)$$
$$= \frac{1}{4} \left(\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2}\right),$$

which proves the equality of the two given expressions.

5. Let $$C = \frac{b}{c}\cos A + \frac{b^2}{2c^2}\cos 2A + \dots$$

$$S = \frac{b}{c}\sin A + \frac{b^2}{2c^2}\sin 2A + \dots$$

$$\therefore C + Si = \frac{b}{c} \cdot e^{Ai} + \frac{b^2}{2c^2} \cdot e^{2Ai} + \dots$$

$$= -\log\left(1 - \frac{b}{c}\, e^{Ai}\right),$$

$$\therefore 1 - \frac{b}{c}(\cos A + i \sin A) = e^{-C-Si} = e^{-C} \cdot e^{-Si}$$

$$= e^{-C}(\cos S - i \sin S).$$

$\therefore$ equating real and unreal parts,

$$e^{-C}\cos S = \frac{c - b\cos A}{c} = \frac{a\cos B}{c}, \quad (1)$$

$$e^{-C}\sin S = \frac{b\sin A}{c} = \frac{a\sin B}{c}, \quad (2)$$

square and add (1) and (2).

$$\therefore e^{-2C} = \frac{a^2}{c^2}, \therefore C = \log\frac{c}{a}.$$

Similarly, if

$$C' = \frac{a\cos B}{c} + \frac{a^2}{2c^2}\cos 2B + \dots$$

$$S' = \frac{a}{c}\sin B + \frac{a^2}{2c^2}\sin 2B + \dots$$

we shall have $$C' = \log\frac{c}{b}$$

$\therefore$ the given expression $= C - C' = \log\frac{c}{a} - \log\frac{c}{b} = \log\frac{b}{a}$.

NOTE.—If we take the value of $C + C'$ we find

$$\frac{b\cos A + a\cos B}{c} + \frac{b^2\cos 2A + a^2\cos 2B}{2c^2} + \dots = \log\frac{c^2}{ab}.$$

Also, if we divide (2) by (1) we get $\tan S = \tan B$.

$$\therefore B = \frac{b}{c}\sin A + \frac{b^2}{2c^2}\sin 2A + \dots$$

6. The angle $HFR = PCD = CQG$; and the angle $FHR = CGQ$.

$\therefore$ the triangles $FHR$ and $CGQ$ are similar.

$$\therefore HR : HF :: CG : GQ$$

$$\therefore GQ \,.\, HR = HF \,.\, CG = \text{constant.}$$

7. The given equation can be put in the form

$$(y - m_1x)(y - m_2x) = 0$$

where $$m_1 + m_2 = -\frac{b}{c}, \; m_1m_2 = \frac{b}{a}.$$

If these two straight lines are at right angles,

$$1 + (m_1 + m_2)\cos\omega + m_1m_2 = 0,$$

$$\therefore \cos\omega = \frac{1 + \frac{b}{a}}{\frac{b}{c}} = c\left(\frac{1}{a} + \frac{1}{b}\right).$$

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## Paper XXIV.

1. Since $\alpha$, $\beta$ are the roots of the 1st equation,

$$\therefore \alpha + \beta = -a, \; \alpha\beta = \tfrac{1}{4}(a^2 - b^2), \; \therefore (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = b^2,$$

$$\therefore \alpha - \beta = \mp b, \; \therefore 2\alpha = -(a \pm b), \; \alpha^2 - \beta^2 = \pm ab,$$

$\therefore$ the equation whose roots are $\alpha + \beta$, $\alpha - \beta$ is

$$x^2 - 2ax + a^2 - \beta^2 = 0,$$

or $$x^2 + (a \pm b)x \pm ab = 0.$$

2. Let $x$ denote the number of feet in a side of the one carpet, $y$ the number in a side of the other. Then we have to find when $ax^2 + \beta y^2$ is a min. subject to the condition $x + y = \text{const.}$

Now $$(ax - \beta y)^2 + \alpha\beta(x + y)^2 = (\alpha + \beta)(ax^2 + \beta y^2);$$

$\therefore ax^2 + \beta y^2$ is a min. when $ax - \beta y = 0$; *i.e.* when $\frac{x}{\beta} = \frac{y}{\alpha}$;

$\therefore$ the areas are as $x^2$ to $y^2$, *i.e.* as $\beta^2$ to $\alpha^2$.

3. The given expression

$$= \tfrac{1}{2}\{\sin(\beta+\gamma-\alpha)\sin 2(\beta-\gamma) + \sin(\gamma+\alpha-\beta)\sin 2(\gamma-\alpha) + \sin(\alpha+\beta-\gamma)\sin 2(\alpha-\beta)\}$$

$$= \tfrac{1}{4}\{\cos(\beta-3\gamma+\alpha) - \cos(3\beta-\gamma-\alpha) + \cos(\gamma-3\alpha+\beta) - \cos(3\gamma-\alpha-\beta) + \cos(\alpha-3\beta+\gamma) - \cos(3\alpha-\beta-\gamma)\};$$

$= 0$, since $\cos(-A) = \cos A$.

4. Let $BAC$ be the given triangle. On $AB$, $AC$ describe two similar isosceles triangles $BDA$, $AEC$, having the angles at their bases each $= \theta$. Let $F$ be the middle point of $BC$.

Then $DF^2 = FB^2 + BD^2 - 2FB \,.\, BD\cos(B+\theta)$,

$EF^2 = FC^2 + CE^2 - 2FC \,.\, CE\cos(C+\theta)$,

and $DF = EF$, $FB = FC$;

$$\therefore BD^2 - CE^2 = 2FB\{BD\cos(B+\theta) - CE\cos(C+\theta)\}.$$

Also $BD = \dfrac{c}{2\cos\theta}$; $CE = \dfrac{b}{2\cos\theta}$; $FB = \dfrac{a}{2}$;

$$\therefore \frac{c^2-b^2}{4\cos^2\theta} = \frac{a}{2\cos\theta}\{c\cos(B+\theta) - b\cos(C+\theta)\},$$

$$= \frac{a}{2\cos\theta}\{c(\cos B\cos\theta - \sin B\sin\theta) - b(\cos C\cos\theta - \sin C\sin\theta)\},$$

$$= \frac{a}{2}(c\cos B - b\cos C), \text{ since } c\sin B = b\sin C,$$

$$= \frac{c^2-b^2}{2},$$

$$\therefore \cos^2\theta = \frac{1}{2}; \quad \therefore \cos\theta = \frac{1}{\sqrt{2}}; \quad \therefore \theta = 45°;$$

$\therefore$ the vertical angles must be right angles.

5. Let the given lines be of lengths $a$, $b$, $c$.

Then $(a+b)^2 - c^2 = (a+b+c)(a+b-c)$.

Construct a rectangle having $GH$, $HK$ for adjacent sides such that $GH = a+b+c$, $GK = a+b-c$. Produce $GH$ to $L$, making $HL = HK$. On $GL$ as diameter describe a semicircle. Produce $KH$ to meet the circumference in $M$. Then the square on $HM$ is easily seen to be the square required.

6. $CAB = 30°$, $DAB = 60°$, $EAB = 90°$, $FAB = 120°$,

$$\therefore AC = AB \cdot \frac{\sin 120°}{\sin 30°} = \sqrt{3} \cdot AB,$$

$$AD = AB \sec 60° = 2AB,$$

$$AE = AC = \sqrt{3}AB,$$

$$AF = AB,$$

$\therefore$ horizontal component

$$= AB + 2AC\cos 30° + 3AD\cos 60° + 4AE\cos 90° + 5AF\cos 120°,$$

$$= AB\left\{1 + 2 \cdot \sqrt{3} \cdot \frac{\sqrt{3}}{2} + 3 \cdot 2 \cdot \frac{1}{2} + 0 + 5\left(-\frac{1}{2}\right)\right\},$$

$$= AB \cdot \frac{9}{2},$$

vertical component

$$= 2AC\sin 30° + 3AD\sin 60° + 4AE + 5AF\sin 30°,$$

$$= AB\left\{2 \cdot \sqrt{3} \cdot \frac{1}{2} + 3 \cdot 2 \cdot \frac{\sqrt{3}}{2} + 4 \cdot \sqrt{3} + 5 \cdot \frac{\sqrt{3}}{2}\right\},$$

$$= AB \cdot \frac{21\sqrt{3}}{2},$$

$\therefore$ resultant $= \frac{AB}{2}\sqrt{81 + 1323}$,

$$= AB\sqrt{351}.$$

7. Let $PG$ be the normal, $Q$ its middle point, $PN$, $QM$ the ordinates of $P$ and $Q$. Then $MN = \frac{1}{2}NG = AS$. $\therefore SM = AN$;

$$\therefore QM^2 = \tfrac{1}{4}PN^2 = AS \cdot AN = AS \cdot SM.$$

$\therefore$ the locus of $Q$ is a parabola, vertex $S$, and latus rectum $\frac{1}{4}$ that of the original parabola.

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## Paper XXV.

1. Let $x - a$ be the common measure. Then if we write $a$ for $x$ in each of the given expressions, they will become $= 0$.

$$\therefore a^2 + aa + b = 0,$$
$$a^2 + a'a + b' = 0;$$
$$\therefore \frac{a^2}{ab' - a'b} = \frac{a}{b - b'} = \frac{1}{a' - a};$$
$$\therefore (ab' - a'b)(a - a') + (b - b')^2 = 0.$$

2. Since $1, x, x^3$ are in H.P.

$$\therefore 1 + \frac{1}{x^3} = \frac{2}{x}; \quad \therefore x^3 - 2x^2 + 1 = 0;$$

$\therefore (x - 1)(x^2 - x - 1) = 0$, and $x - 1 \neq 0$; $\therefore x^2 - x = 1 \ldots (A)$

Since $1, y^2, y^3$ are in H.P.

$$\therefore 1 + \frac{1}{y^3} = \frac{2}{y^2}; \quad \therefore y^3 - 2y + 1 = 0;$$

$\therefore (y - 1)(y^2 + y - 1) = 0$, and $y - 1 \neq 0$; $\therefore y^2 + y = 1 \ldots (B)$

Subtracting $(B)$ from $(A)$ we have $x^2 - y^2 - (x + y) = 0$;

$\therefore (x + y)(x - y - 1) = 0$, and $x + y \neq 0$; $\therefore x - y = 1 \ldots (C)$

$$\therefore y + y^2 = 1 = x - y = x^2 - x;$$

$\therefore -y^2, y, x, x^2$ are in A.P. the common difference being $= 1$.

Again from $(C)$, $x^2 - xy = x$, and $y^2 = 1 - y$;

$$\therefore x^2 - xy + y^2 = x - y + 1;$$
$$\therefore -y^2 + y + x + x^2 = (x + y)(x - y + 1)$$
$$= (x + y)(x^2 - xy + y^2)$$
$$= x^3 + y^3.$$

3. The given expression

$$= \cos\frac{B+C}{2}\cos\frac{B-C}{2} + \cos\frac{C+A}{2}\cos\frac{C-A}{2} + \cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$= \tfrac{1}{2}\{\cos B + \cos C + \cos C + \cos A + \cos A + \cos B\}$$
$$= \cos A + \cos B + \cos C.$$

4. $$2\cot 2\theta = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} = \cot\theta - \tan\theta;$$

$$\therefore \tan\theta = \cot\theta - 2\cot 2\theta;$$

$$\frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2} - \cot\theta.$$

So $$\frac{1}{4}\tan\frac{\theta}{4} = \frac{1}{4}\cot\frac{\theta}{4} - \frac{1}{2}\cot\frac{\theta}{2};$$

$$\therefore \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{4}\tan\frac{\theta}{4} = \frac{1}{4}\cot\frac{\theta}{4} - \cot\theta.$$

By proceeding in the same manner, and dividing by $2^2$

$$\frac{1}{2^3}\tan\frac{\theta}{2^3} + \frac{1}{2^4}\tan\frac{\theta}{2^4} = \frac{1}{2^4}\cot\frac{\theta}{2^4} - \frac{1}{2^2}\cot\frac{\theta}{2^2};$$

$$\frac{1}{2^5}\tan\frac{\theta}{2^5} + \frac{1}{2^6}\tan\frac{\theta}{2^6} = \frac{1}{2^6}\cot\frac{\theta}{2^6} - \frac{1}{2^4}\cot\frac{\theta}{2^4};$$

. . . .

$$\frac{1}{2^{2n-1}}\tan\frac{\theta}{2^{2n-1}} + \frac{1}{2^{2n}}\tan\frac{\theta}{2^{2n}} = \frac{1}{2^{2n}}\cot\frac{\theta}{2^{2n}} - \frac{1}{2^{2n-2}}\cot\frac{\theta}{2^{2n-2}};$$

$\therefore$ by addition

$$\cot\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \ldots + \frac{1}{2^{2n}}\tan\frac{\theta}{2^{2n}} = \frac{1}{2^{2n}}\cot\frac{\theta}{2^{2n}}$$

$$= \frac{1}{\theta}\cdot\frac{\frac{\theta}{2^{2n}}}{\sin\frac{\theta}{2^{2n}}}\cdot\cos\frac{\theta}{2^{2n}} = \frac{1}{\theta} \text{ when } n \text{ is very large.}$$

5. Let $A$ be the centre of the circle to which the tangents are drawn, $B$ the centre of the other circle. Let the circle, centre $B$, cut $PQ$ in $R$ and $PQ'$ produced in $S$. Then $AP$ bisects the angle $QPQ'$, or $RPS$. $\therefore$ the arc $AR$ = arc $AS$. $\therefore$ $RS$ is perpendicular to $AB$, and $\therefore$ fixed in direction.

Join $QC$, $CQ'$. Then the angle $Q'CA$ = supplement of $ASP$ = $ARP$ = supplement of $ARQ$ = supplement of $ACQ$.

$\therefore$ $ACQ + ACQ' = 2$ right angles. $\therefore$ $QC$ and $CQ'$ are in one straight line. $\therefore$ $AB$, $RS$, and $QQ'$ are concurrent.

6. Tripos 1875. Tuesday morning. No. 4.

7. Since $y = x\tan\alpha$ and $y = x\tan\beta$ are at right angles.

$$\therefore 1 + (\tan\alpha + \tan\beta)\cos\omega + \tan\alpha\tan\beta = 0,$$

$$\therefore \cos\omega = -\frac{1 + \tan\frac{11\pi}{24}\tan\frac{19\pi}{24}}{\tan\frac{11\pi}{24} + \tan\frac{19\pi}{24}} = -\frac{\cos\frac{19-11}{24}\pi}{\sin\frac{19+11}{24}\pi} = -\frac{\cos\frac{\pi}{3}}{\sin\frac{5\pi}{4}}$$

$$= \frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{4}} = \frac{1}{\sqrt{2}}, \therefore \omega = \frac{\pi}{4}.$$

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## Paper XXVI.

1. By ordinary division we find

$$\frac{1}{19} = \cdot\dot{0}52631578947368421\dot{}.$$

By LIX. 1, we know that each of the decimals $\frac{1}{19}, \frac{2}{19}, \cdots \frac{18}{19}$ contains the same 18 digits in the same cyclical order, and as the right hand figure of any one of the decimals is formed by multiplying the numerator of the decimal by unity, the other digits can be written down from right to left by inspection. Thus

$$\frac{7}{19} = \cdot\dot{3}68421052631578947\dot{},$$

$$\frac{17}{19} = \cdot\dot{8}94736842105263157\dot{}.$$

2. The given expression

$$= w^3(x+y+z) + 2w^2(x^2+y^2+z^2) + w(x^3+y^3+z^3) + w^2(yz+zx+xy)$$
$$- 6xyzw + xyz(x+y+z) + 4xyzw$$
$$= -w^4 + 2w^2(x^2+y^2+z^2) + w^2(yz+zx+xy) + w(x^3+y^3+z^3-3xyz)$$
$$= -w^4 + w^2\{2(x^2+y^2+z^2) + yz+zx+xy\} - w^2(x^2+y^2+z^2-yz-zx-xy)$$
$$= -w^4 + w^2\{(x^2+y^2+z^2) + 2(yz+zx+xy)\}$$
$$= -w^2\{w^2 - (x+y+z)^2\} = 0.$$

3. Expression $= \dfrac{\dfrac{1}{2\sin x \sin y}\left(\dfrac{1}{\cos x} - \dfrac{1}{\cos y}\right)}{\dfrac{1}{2\sin x \sin y}\left(\dfrac{1}{\cos x} + \dfrac{1}{\cos y}\right)} = \dfrac{\cos y - \cos x}{\cos y + \cos x}$

$$= \frac{2\sin\frac{x+y}{2}\sin\frac{x-y}{2}}{2\cos\frac{x+y}{2}\cos\frac{x-y}{2}} = \tan\frac{x+y}{2}\tan\frac{x-y}{2}.$$

4. Let $\theta$ denote the circular measure of the angle $143^\circ\ 14'\ 22''$.

$$\therefore \theta = \frac{10}{4} = \frac{5}{2};$$

$$\therefore \pi = \frac{180^\circ}{143^\circ\ 14'\ 22''}.\theta = \frac{810000}{257831} = 3{\cdot}1415\ldots$$

5. Let $A$, $A'$ be the vertices of the given section. Then since $CAD$, $CA'D$ are right angles, the sphere passes through $A$ and $A'$. Now the centre of the section of the sphere made by the given plane will be the intersection of $CD$ with $AA'$, and will $\therefore$ coincide with the centre of the conic. $\therefore$ the plane intersects the sphere in the auxiliary circle.

6. Let $O$ be the centre of the circle, $OD$ the perpendicular. Produce $DO$ to meet the sphere in $D'$. Let $r$, $R$ be the radii of the circle and sphere. Then $OD = r\sqrt{2}$, $OD' = 2R - OD = 2R - r\sqrt{2}$.

Since $DD'$ is a diameter of the sphere, $DAD'$ is a right angle, and $DA^2 = DO^2 + OA^2$, since $DO$ is perpendicular to $AO$;

and
$$D'A^2 = D'O^2 + OA^2.$$

$$\therefore 4R^2 = DD'^2 = DA^2 + D'A^2$$
$$= 2r^2 + r^2 + (2R - \sqrt{2}r)^2 + r^2$$
$$= 6r^2 + 4R^2 - 4\sqrt{2}Rr.$$
$$\therefore 4\sqrt{2}\,.\,R\,.\,r = 6r^2 = 3\sqrt{2}r\,.\,\sqrt{2}r,$$
$$\therefore 4R = 3\sqrt{2}r = 3\,.\,OD \therefore R = \tfrac{3}{4}\,OD.$$

7. Let $CD$ be the equi-conjugate diameter to which $PM$ is perpendicular. Equation to $CD$ is $\dfrac{x}{a} = \dfrac{y}{b}$. $\therefore$ if $x'y'$ be coordinates of $P$, the equation to $PM$ is

$$b(y - y') + a(x - x') = 0.$$

The coordinates of $M$ obtained from these equations are

$$\frac{a}{a^2+b^2}(by'+ax'),\ \frac{b}{a^2+b^2}(by'+ax').$$

Similarly, the coordinates of $N$ are

$$-\frac{a}{a^2+b^2}(by'-ax'),\ \frac{b}{a^2+b^2}(by'-ax').$$

$\therefore$ if $\xi, \eta$ be the coordinates of $Q$, the middle point of $MN$,

$$\xi = \frac{a^2x'}{a^2+b^2},\ \eta = \frac{b^2y'}{a^2+b^2}.$$

Now the equation to the tangent at $P$ is $\frac{yy'}{a^2}+\frac{xx'}{b^2}=1$.

$\therefore$ „ „ normal „ $(y-y')\frac{x'}{b^2} = (x-x')\frac{y'}{a^2}$,

and we see by trial that the values of $\xi, \eta$ satisfy this equation. $\therefore$ the normal at $P$ passes through $Q$.

*This question may also be solved geometrically.

Let $Cx$, $Cy$ be the equiconjugate diameters, and let the tangent at $P$ meet them in $T$, $t$, and bisect $Tt$ in $Q$.

Then the angle $PCT = QCt$. (See solution to XLVIII. No. 7).

Draw $PN$, $PM$, $PU$ perpendicular to $Cx$, $Cy$, $Tt$ respectively. Produce $CQ$ to $L$ so that $CQ = QL$, and complete the parallelogram $CTLt$. Then, in the triangles $PMN$, $CtL$ we have $PM$ perpendicular to $Ct$, and $PN$ perpendicular to $CT$, and $\therefore$ also to $tL$, and the angle $PMN = PCT = QCt$, and $PNM = PCt = QCT = CLt$. $\therefore$ the triangles $PMN$ and $CtL$ are similar, the homologous sides being perpendicular. Now, in the triangle $CtL$, $tQ$ bisects $CL$, $\therefore$ in $PMN$, $PU$ (which is perpendicular to $tQ$) bisects $MN$.

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## Paper XXVII.

1. Let $\left.\begin{aligned} a &= a+(p-1)\beta \\ b &= a+(q-1)\beta \\ c &= a+(r-1)\beta \end{aligned}\right\}(A)$ and $\left.\begin{aligned} a &= \gamma.\rho^{p-1} \\ b &= \gamma.\rho^{q-1} \\ c &= \gamma.\rho^{r-1} \end{aligned}\right\}(B)$

From $(A)$ $a-b=(p-q)\beta$, $b-c=(q-r)\beta$, $c-a=(r-p)\beta$.

$\therefore a^{b-c} = \gamma^{(q-r)\beta}.\rho^{(p-1)(q-r)\beta}$, and similar expressions for $b^{c-a}$ and $c^{a-b}$. $\therefore$ in the required product,

index of $\gamma = (q - r + r - p + p - q)\beta = 0$,

index of $\rho = \{(p-1)(q-r) + (q-1)(r-p) + (r-1)(p-q)\}\beta = 0$.

$\therefore$ given expression $= \gamma^\circ . \rho^\circ = 1$.

2. Divide both equations by $y^3$, and write $z$ for $\frac{x}{y}$. We get

$$az^3 + bz^2 + cz + d = 0 \quad (1)$$
$$a'z^3 + b'z^2 + c'z + d' = 0 \quad (2).$$

Multiply (1) by $a'$, (2) by $a$, and subtract.

$\therefore z^2(a'b - ab') + z(a'c - ac') + a'd - ad' = 0$, or $az^2 + \beta z + \gamma = 0$.

Multiply (1) by $d'$, (2) by $d$, subtract, and divide by $z$

$\therefore z^2(a'd - ad') + z(b'd - bd') + c'd - cd' = 0$, or $a'z^2 + \beta' z + \gamma' = 0$.

$$\therefore \frac{z^2}{\beta\gamma' - \beta'\gamma} = \frac{z}{\gamma a' - \gamma' a} = \frac{1}{a\beta' - a'\beta}$$
$$\therefore (a\beta' - a'\beta)(\beta\gamma' - \beta'\gamma) = (\gamma a' - \gamma' a)^2,$$

and writing for $a$, $a'b - ab'$ &c., we obtain the required result, multiplied by a factor which is easily seen to be $a'd - ad'$.

For multiplying out we have

$$a\beta\beta'\gamma' - a\beta'^2\gamma - a'\beta^2\gamma' + a'\beta\beta'\gamma = a'^2\gamma^2 - 2aa'\gamma\gamma' + a^2\gamma'^2.$$

Now remembering that $a' = \gamma = a'd - ad'$, we see that this result may be written

$$\begin{aligned} a'(\beta\beta'\gamma - a\beta'^2 - \beta^2\gamma' - a'\gamma^2 + 2a'\gamma\gamma') &= a\gamma'(a\gamma' - \beta\beta') \\ &= a\gamma'\{(a'b - ab')(c'd - cd') - (a'c - ac')(b'd - bd')\} \\ &= a\gamma'(a'bc'd - a'b'cd - abc'd' + ab'cd') \\ &= a\gamma'(a'd - ad')(bc' - b'c) \\ &= aa'\gamma'(bc' - b'c). \end{aligned}$$

$\therefore$ dividing out by $a'$ we obtain the result required.

3. $$\sec\theta \sec 2\theta = \frac{1}{\cos\theta(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}$$
$$\equiv \frac{A}{\cos\theta} + \frac{B}{\cos\theta - \sin\theta} + \frac{C}{\cos\theta + \sin\theta}$$

suppose, where $A$, $B$, and $C$ are numerical constants.

$\therefore 1 \equiv A(\cos^2\theta - \sin^2\theta) + \cos\theta\{B(\cos\theta + \sin\theta) + C(\cos\theta - \sin\theta)\}$.

Since this is true whatever be the value of $\theta$, let $\cos\theta = 0$,

$\therefore 1 = -A$. Then by subtraction we have

$$0 \equiv 2\cos^2\theta \,.\, A + \cos\theta \left\{B(\cos\theta + \sin\theta) + C(\cos\theta - \sin\theta)\right\}$$
$$\equiv 2\cos\theta \,.\, A + B(\cos\theta + \sin\theta) + C(\cos\theta - \sin\theta).$$

Since this is always true, let $\cos\theta = \sin\theta$,

$$\therefore 0 = 2\cos\theta \,.\, A + 2\cos\theta \,.\, B = A + B, \quad \therefore B = -A = 1.$$

Again, let $\cos\theta = -\sin\theta$,

$$\therefore 0 = 2\cos\theta \,.\, A + 2\cos\theta \,.\, C = A + C, \quad \therefore C = -A = 1$$

$$\therefore \sec\theta \sec 2\theta = \frac{-1}{\cos\theta} + \frac{1}{\cos\theta + \sin\theta} + \frac{1}{\cos\theta - \sin\theta}.$$

4. Let $x$ denote no. degrees in $A$. Then $\frac{9}{10}x$ and $\frac{180x}{\pi}$ are the no. of degrees in $B$ and $C$ respectively.

$$\therefore x + \frac{9}{10}x + \frac{180}{\pi} \,.\, x = 180;$$

$$\therefore x = \frac{1800\pi}{19\pi + 1800}.$$

$\therefore$ the angles of the triangle in degrees are

$$\frac{1800\pi}{19\pi + 1800}, \quad \frac{1620\pi}{19\pi + 1800}, \quad \frac{324000}{19\pi + 1800},$$

*i.e.* 3° 2′ 26″.3, 2° 44′ 11″.67, 174° 13′ 22″.03 approximately.

5. I. Geometrical. Let the straight lines $TFP$, $FRF'$, $TF'Q$ touch a parabola in $P, R, Q$, and let the middle tangent $FRF'$ meet the directrix in $D$. Draw $TK$ perpendicular to $FF'$, meeting $SD$ in $N$, and the directrix in 0. Then since $RSD$ is a right angle, and $FF'T = F'SR$, $\therefore F'TN = F'SN$. $\therefore N$ is on the circum-circle of $PQR$.

Again, $FD$ bisects the angle $ODS$, and $\therefore$ it bisects $ON$, to which it is at right angles. $\therefore OK = KN$.

Now in any triangle if a perpendicular be drawn from an angle to the opposite side, the portion intercepted between the orthocentre and the circum-circle of the triangle is bisected by the side of the triangle.

$\therefore O$ is the orthocentre of $FTF'$, and lies on the directrix.

II. Analytical. Let the same letters be employed as in the former proof. Let $y^2 = 4ax$ be the equation to the parabola,

$$(x_1y_1), \ (x_2y_2), \ x_3y_3),$$

the coordinates of $P, R, Q$.

Then equation to $PT$ is $yy_1 = 2a(x + x_1)$; (1)

" " $FF'$ " $yy_2 = 2a(x + x_2)$; (2)

" " $TQ$ " $yy_3 = 2a(x + x_3)$; (3).

From (2) and (3) we have

$$y(y_2 - y_3) = 2a(x_2 - x_3) = \tfrac{1}{2}(y_2^2 - y_3^2),$$

$\therefore$ coordinates of $F'$ are $\eta = \dfrac{y_2 + y_3}{2}$, $\xi = \dfrac{y_2 y_3}{4a}$.

Draw $F'E'$, $TE$ perpendicular to the opposite sides.

Then equation to $F'E'$ is $\left(y - \dfrac{y_2 + y_3}{2}\right)2a + \left(x - \dfrac{y_2 y_3}{4a}\right)y_1 = 0$,

$$\text{or } 2ay + xy_1 = a(y_2 + y_3) + \frac{y_1 y_2 y_3}{4a}.$$

Similarly " $TE$ is $2ay + xy_2 = a(y_3 + y_1) + \dfrac{y_1 y_2 y_3}{4a}$.

$\therefore$ at $O$ we have $x(y_1 - y_2) = a(y_2 - y_1)$ $\therefore x = -a$ $\therefore$ $O$ is a point on the directrix.

III. By reciprocal polars.

To the points of contact of two tangents through the origin correspond the tangents at the two points at infinity on the reciprocal curve, *i.e.* the asymptotes, which will contain an angle equal to that between the two tangents from the origin to the original curve. Conversely, if we reciprocate a rectangular hyperbola with respect to any point $O$ on the curve, we shall obtain a parabola whose directrix will pass through $O$. Also, if a triangle be reciprocated with respect to its orthocentre the reciprocal triangle will have the same orthocentre. $\therefore$ reciprocating the known theorem, 'The orthocentre of a triangle inscribed in a rectangular hyperbola lies on the curve,' we have, 'The orthocentre of a triangle circumscribing a parabola lies on the directrix.'

IV. By Brianchon's Theorem.

Let $FTF'$ be the triangle whose sides touch the parabola at $P$, $R$, $Q$, as in solutions I. and II. Let $PF$, $FR$ meet the directrix in $A$ and $B$, and draw the tangents $AP'$, $BR'$. Then remembering that the line at infinity touches the parabola, we have six tangents which form a hexagon, and by Brianchon's Theorem we know that the three opposite diagonals are concurrent. Now these diagonals are (1) the line joining $T$ with the point where $BR'$ intersects the line at infinity, *i.e.* a line through $T$ parallel to $BR'$, and $\therefore$ perpendicular to $FF'$, (2) the line joining $F'$ with the point where $AP'$ meets the line at

infinity, *i.e.* a line through $F'$ parallel to $AP'$, and $\therefore$ perpendicular to $TF$, and (3) $AB$ which is the directrix. $\therefore$ the directrix passes through the intersection of (1) and (2), *i.e.* through the orthocentre of the triangle $FTF'$.

6. NOTE.—The paper is supposed to be on the point of falling over. Let $A$ be the vertex, $AB$, $AC$ the sides, of the triangle formed when the corners are doubled over. From $A$ draw a straight line $AF$ parallel to a side of the rectangle, dividing the paper into two equal parts. Let the edge of the table $EF$ cut this line in $L$, and let $BC$ cut it in $K$. Then in calculating the weights of the parts we need only consider lengths along $AF$.

Let $AF = a$. Then $BC = \frac{a}{\sqrt{2}}$. Let $x = AL$ = length of part on the table. Let $G$, $G_1$, $G_2$, $G_3$ be the centres of gravity of $ACDEB$, $ABC$, $BCDE$, $EDF$ respectively.

Then $$KG_1 = \tfrac{1}{3}AK = \tfrac{1}{3} \cdot \frac{a}{2\sqrt{2}};$$

$$\therefore LG_1 = LK + KG_1 = x - \frac{a}{2\sqrt{2}} + \frac{a}{6\sqrt{2}} = x - \frac{a}{3\sqrt{2}};$$

$$KG_2 = \tfrac{1}{2}(AL - AK) = \tfrac{1}{2}\left(x - \frac{a}{2\sqrt{2}}\right) = LG_2.$$

Taking moments about $L$

$$LG \cdot LA = LG_2 \cdot LK + LG_1 \cdot AK;$$

$$\therefore LG \cdot x = \tfrac{1}{2}\left(x - \frac{a}{2\sqrt{2}}\right)^2 + \frac{a}{2\sqrt{2}}\left(x - \frac{a}{3\sqrt{2}}\right) = \frac{x^2}{2} - \frac{a^2}{48};$$

$$\therefore \frac{x^2}{2} - \frac{a^2}{48} = LG \cdot x = LG_3 \cdot LF = \tfrac{1}{2}(a - x)^2;$$

$$\therefore ax = \frac{a^2}{48} + \frac{a^2}{2} = \frac{25}{48}a^2; \quad \therefore x = \frac{25}{48} \cdot a.$$

7. Let the equation to the ellipse referred to conjugate diameters be $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$, and let $x'y'$ be coordinates of any point on it.

Then if $A$, $B$ be the ends of the major and minor axes,

$$AB^2 = a^2 + b^2 = a'^2 + b'^2.$$

Then we have to show that $a^2 + b^2$ is not less than $(x' + y')^2$,

*i.e.* $a'^2 + b'^2 - (x'^2 + y'^2 + 2x'y')$ is positive,

*i.e.* $a'^2 - x'^2 + b'^2 - y'^2 - 2x'y'$ is positive,

*i.e.* $a'^2\left(1 - \frac{x'^2}{a'^2}\right) + b'^2\left(1 - \frac{y'^2}{b'^2}\right) - 2x'y'$ is positive,

*i.e.* $\frac{a'^2 \cdot y'^2}{b'^2} + \frac{b'^2 \cdot x'^2}{a'^2} - 2x'y'$ is positive.

*i.e.* $\left(\frac{a'y'}{b'} - \frac{b'x'}{a'}\right)^2$ is positive, which we know to be the case.

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## Paper XXVIII.

1. The com. dif. of the A.P. is $b - a$,

$$\therefore (n + 2)^{\text{th}} \text{ term} = (n + 1)b - na.$$

The com. ratio of the G.P. is $\frac{b}{a}$,

$$\therefore (n + 2)^{\text{th}} \text{ term} = \frac{b^{n+1}}{a^n}.$$

The com. dif. of an A.P. in which $\frac{1}{a}, \frac{1}{b}$ are first two terms is $\frac{1}{b} - \frac{1}{a}$,

$$\therefore (n + 2)^{\text{th}} \text{ term of this A.P. is } \frac{1}{a} + (n + 1)\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{(n+1)a - nb}{ab}$$

$$\therefore (n + 2)^{\text{th}} \text{ term of the H.P. is } \frac{ab}{(n + 1)a - nb}.$$

Since the three terms thus found are in G.P.

$$\therefore \frac{b^{2n+2}}{a^{2n}} = \{(n + 1)b - na\} \frac{ab}{(n + 1)a - nb};$$

$$\therefore b^{2n+1}\{(n + 1)a - nb\} = a^{2n+1}\{(n + 1)b - na\};$$

$$\therefore ab(n + 1)(b^{2n} - a^{2n}) = n(b^{2n+2} - a^{2n+2})$$

which gives us the required result.

2. If $n = 3$, we have when

$$(a_1^2 + a_2^2)(a_2^2 + a_3^2) = (a_1a_2 + a_2a_3)^2;$$

$$\therefore a_1^2a_2^2 + a_1^2a_3^2 + a_2^4 + a_2^2a_3^2 = a_1^2a_2^2 + a_2^2a_3^2 + 2a_1a_2^2a_3;$$

$$\therefore (a_1a_3 - a_2^2)^2 = 0;$$

$\therefore$ when the given condition holds, we have $a_1$, $a_2$, $a_3$ in G.P. By ordinary multiplication we may show that with the given condition the theorem holds when $n = 4$. Suppose now that it is true, that with the given condition, when $n$ has any particular value the $n$ quantities are in G.P. and suppose the given condition to hold between $n + 1$ quantities. Then we shall show that these $n + 1$ quantities will be in G.P. For by the given condition

$$(a_1^2 + a_2^2 + \ldots + a^2_{n-1} + a^2_n)(a_2^2 + a_3^2 + \ldots + a^2_n + a^2_{n+1}) = (a_1a_2 + \ldots + a_{n-1}a_n + a_na_{n+1})^2;$$

$$\therefore (a_1^2+a_2^2+\ldots+a^2_{n-1})(a_2^2+a_3^2+\ldots+a^2_n)+a^2_n(a_2^2+a_3^2+\ldots+a^2_n) + a^2_{n+1}(a^2_1+a^2_2+\ldots+a^2_n) = (a_1a_2+\ldots+a_{n-1}a_n)^2+2a_na_{n+1}(a_1a_2+\ldots+a_{n-1}a_n)+a^2_na^2_{n+1}.$$

Now $(a_1^2+a_2^2+\ldots+a^2_{n-1})(a_2^2+a_3^2+\ldots+a^2_n)=(a_1a_2+\ldots+a_{n-1}a_n)^2$;

$$\therefore (a_1a_{n+1}-a_2a_n)^2+(a_2a_{n+1}-a_3a_n)^2+\ldots+(a_n^2-a_{n-1}a_{n+1})^2=0;$$

$\therefore$ each of the expressions within the brackets must $= 0$;

$$\therefore \frac{a_{n+1}}{a_n} = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots = \frac{a_n}{a_{n-1}}.$$

Now we know that the first $n$ quantities are in G.P., and we see that if we multiply $a_n$ by the common ratio of the G.P. we get $a_{n+1}$. $\therefore$ if the theorem is true for $n$ terms, it is also true for $n + 1$. Now we know that it is true when $n = 3$, and when $n = 4$. $\therefore$ it is true universally.

3. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$. Todh. *Trig.* Art. 114. And $\sin \frac{A}{2}$, $\sin \frac{B}{2}$, $\sin \frac{C}{2}$ are all positive. $\therefore$ expression $> 1$.

Again, by Todh. *Trig.* p. 157, No. 40

$8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} < 1$, except when $A = B = C$, in which case it $= 1$,

$$\therefore \cos A + \cos B + \cos C \not> \tfrac{3}{2}.$$

4. $x \sin^2 A \cos B - y \sin^2 B \cos A + z(\cos^2 A - \cos^2 B) = 0,$

$x \sin^2 A \cos C + y(\cos^2 A - \cos^2 C) - z \sin^2 C \cos A = 0,$

$\therefore$ eliminating $x$ we have

$$\begin{aligned} 0 &= y\{\cos B\,(\cos^2 A - \cos^2 C) + \sin^2 B \cos A\} \\ &\quad - z\{\sin^2 C \cos A \cos B - \cos C(\cos^2 A - \cos^2 B)\}, \\ &= y(\cos A - \cos B \cos C)\,(\cos C + \cos A \cos B) \\ &\quad - z(\cos A - \cos B \cos C)\,(\cos B + \cos C \cos A), \\ &= y(\cos C + \cos A \cos B) - z(\cos B + \cos C \cos A), \\ &= by - cz. \quad \text{Todh. } \textit{Trig.} \text{ p. 156. No. 24.} \end{aligned}$$

$\therefore by = cz = k$ suppose

$$\begin{aligned} \therefore x \sin^2 A \cos B &= k\left(\frac{\sin^2 B \cos A}{b} + \frac{\cos^2 B - \cos^2 A}{c}\right), \\ &= k\left(\frac{\sin^2 B \cos A}{b} + \frac{\sin \overline{A + B} \sin \overline{A - B}}{c}\right), \\ &= k \frac{\sin A}{a} (\sin B \cos A + \sin \overline{A - B}), \\ &= k \frac{\sin A}{a} \sin A \cos B, \end{aligned}$$

$$\therefore ax = k = by = cz.$$

5. Let the perpendiculars from $D$ meet the sides $BC$, $CA$, $AB$ respectively in $F$, $G$, $H$, and let those from $E$ meet the same sides in $F'$, $G'$, $H'$. Then the angle

$$\begin{aligned} FPF' &= HPG' = HH'P + H'HP = BEF' + HAG' + PGA \\ &= BEF' + BAC + CDF = \text{const.} \end{aligned}$$

$\therefore$ the locus of $P$ is a segment of a circle on $FF'$.

6. Let $S$ be a fixed point on the circumference of a circle, and $AB$ a chord which subtends half a right angle at $S$. Then it is clear that $AB$ always touches a circle concentric with the given circle, and that the tangents to it from $S$ are at right angles. By reciprocating this we obtain the required theorem.

7. The equation to the pair of tangents through $(h, k)$ is

$$(x^2 + y^2 - c^2)\,(h^2 + k^2 - c^2) = (xh + yk - c^2)^2.$$

The two straight lines through the origin parallel to these tangents are obtained by equating to zero the terms of highest dimension in the above equation, viz.

$$y^2(h^2 - c^2) - 2hkxy + x^2(k^2 - c^2) = 0.$$

If $m_1$, $m_2$ be the tangents of the angles which these lines make with the axis of $x$,

$$m_1 + m_2 = \frac{2hk}{h^2 - c^2},\ m_1 m_2 = \frac{k^2 - c^2}{h^2 - c^2}.$$

If the two lines are at right angles we must have

$$\begin{aligned} 0 &= 1 + (m_1 + m_2)\cos\omega + m_1 m_2, \\ &= 1 + \frac{2hk\cos\omega}{h^2 - c^2} + \frac{k^2 - c^2}{h^2 - c^2}, \\ &= \frac{h^2 + k^2 + 2hk\cos\omega - 2c^2}{h^2 - c^2}, \end{aligned}$$

$\therefore$ $(h, k)$ lies on the circle

$$x^2 + y^2 + 2xy\cos\omega = 2c^2.$$

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## Paper XXIX.

1. $x_3 = \log_{x_1} x_2$; $\therefore x_1^{x_3} = x_2$; $x_2^{x_4} = x_3$; .... $x_{n-2}^{x_n} = x_{n-1}$;

$$x_{n-1}^{x_1} = x_n;\ x_n^{x_2} = x_1.$$

Now $x_1^{x_3} = x_2$, $\therefore x_1^{x_3 x_4} = x_2^{x_4} = x_3$; $\therefore x_1^{x_3 x_4 x_5} = x_3^{x_5} = x_4$, &c.

$$\therefore x_1^{x_2 x_4 \ldots x_{n-1}} = x_{n-2},\ \therefore x_1^{x_3 \ldots x_n} = x_{n-2}^{x_n} = x_{n-1};$$

$$\therefore x_1^{x_3 \ldots x_n . x_1} = x_{n-1}^{x_1} = x_n;$$

$$\therefore x_1^{x_3 \ldots x_n . x_1 x_2} = x_n^{x_2} = x_1;$$

$$\therefore x_1 x_2 \ldots x_n = 1.$$

2. $a^2x^2 + b^2y^2 + c^2z^2 = 0$; $a^2x^3 + b^2y^3 + c^2z^3 = 0$;

$$\frac{1}{x} - a^2 = \frac{1}{y} - b^2 = \frac{1}{z} - c^2 = R \text{ suppose,}$$

$$\therefore 1 - a^2x = Rx;\ 1 - b^2y = Ry;\ 1 - c^2z = Rz.$$

$$\begin{aligned} \therefore a^2x^2 + b^2y^2 + c^2z^2 - (a^4x^3 + b^4y^3 + c^4z^3) \\ = R(a^2x^3 + b^2y^3 + c^2z^3) = 0, \end{aligned}$$

$$\therefore a^4x^3 + b^4y^3 + c^4z^3 = a^2x^2 + b^2y^2 + c^2z^2 = 0.$$

Also $$a^4x^2 + b^4y^2 + c^4z^2 - (a^6x^3 + b^6y^3 + c^6z^3)$$
$$= R(a^4x^3 + b^4y^3 + c^4z^3) = 0,$$
$$\therefore\ a^6x^3 + b^6y^3 + c^6z^3 = a^4x^2 + b^4y^2 + c^4z^2.$$

3. $$b^2\cos 2B + 2bc\cos(B - C) + c^2\cos 2C,$$
$$= b^2(\cos^2B - \sin^2B) + 2bc(\cos B\cos C + \sin B\sin C) + c^2(\cos^2C - \sin^2C)$$
$$= (b\cos B + c\cos C)^2 - (b\sin B - c\sin C)^2.$$

Now $b = c\cos A + a\cos C$; $c = a\cos B + b\cos A$,

$$\therefore\ b\cos B + c\cos C = 2a\cos B\cos C + \cos A(c\cos B + b\cos C),$$
$$= 2a\cos B\cos C + a\cos A = a\cos(B - C);$$

$$b\sin B - c\sin C = \frac{a}{\sin A}(\sin^2B - \sin^2C),$$

$$= \frac{a}{\sin A}\sin(B + C)\sin(B - C) = a\sin(B - C);$$

$$\therefore \text{ given exp.} = a^2\{\cos^2(B - C) - \sin^2(B - C)\},$$
$$= a^2\cos 2(B - C).$$

4. Let $$S = \sin^2x - \tfrac{1}{2}\sin^22x + \ldots.$$
$$C = \cos^2x - \tfrac{1}{2}\cos^22x + \ldots.$$
$$\therefore\ C + S = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots. = \log(1 + 1) = \log 2.$$

Let $$C' = C - S = \cos 2x - \tfrac{1}{2}\cos 4x + \tfrac{1}{3}\cos 6x \ldots.$$

and $$S' = \sin 2x - \tfrac{1}{2}\sin 4x + \tfrac{1}{3}\sin 6x \ldots.$$
$$\therefore\ C' + S'i = e^{2xi} - \tfrac{1}{2}e^{4xi} + \tfrac{1}{3}e^{6xi} \ldots = \log(1 + e^{2xi}),$$
$$\therefore\ 1 + e^{2xi} = e^{C'}.e^{S'i},$$
$$\therefore\ 1 + \cos 2x + i\,.\sin 2x = e^{C'}(\cos S' + i\sin S').$$

$\therefore$ equating real and unreal parts,

$e^{C'}\cos S' = 1 + \cos 2x$; $e^{C'}\sin S' = \sin 2x$. Square and add.

$\therefore e^{2C'} = 2(1 + \cos 2x) = 4\cos^2x$ $\therefore e^{C'} = 2\cos x$ $\therefore C' = \log 2 + \log\cos x$.

$$\therefore C - S = C' = \log 2 - \log\sec x, \text{ and } C + S = \log 2,$$
$$\therefore 2S = \log\sec x,\ \therefore S = \tfrac{1}{2}\log\sec x.$$

5. Let $A$ be the vertex, $BCD$ the base of the tetrahedron. Let the angles $CAD$, $DAB$, $BAC$ be bisected by the lines $AF$, $AG$, $AE$, which cut $CD$, $DB$, $BC$ respectively in the points $F$, $G$, $E$.

Then since $AF$ bisects the angle $DAC$,

$$\therefore DF : FC :: DA : AC.$$

Similarly $$CE : EB :: CA : AB,$$

and $$BG : GD :: BA : AD,$$

$$\therefore DF . CE . BG = FC . EB . GD,$$

$\therefore$ $BF$, $CG$ and $DE$ are concurrent.

6. If a triangle be described about a parabola, its orthocentre will lie on the directrix. Therefore, if a parabola be described touching the four straight lines, the orthocentres of the four triangles formed by them will lie on the directrix.

7. Let $P$ $(x'y')$ be the point on the ellipse from which the tangents are drawn to the circle. The polar of $P$ with respect to the circle is

$$xx' + yy' = c^2. \quad \therefore \alpha = \frac{c^2}{x'}, \beta = \frac{c^2}{y'}.$$

$$\therefore \frac{1}{a^2\alpha^2} + \frac{1}{b^2\beta^2} = \frac{1}{c^4}\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right) = \frac{1}{c^4},$$

since $(x'y')$ is on the ellipse.

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## Paper XXX.

1. Tripos 1878. Monday afternoon. No. 4.

2. (1) $b^3 = x^3 + y^3 = (x+y)(x^2+y^2-xy) = (x+y)\{(x+y)^2 - 3xy\}$,

$$\therefore \frac{b^3}{x+y} - (x+y)^2 = -3xy = 3a(x+y) - 3ab.$$

$$\therefore (x+y)^3 + 3a(x+y)^2 - 3ab(x+y) - b^3 = 0,$$

$$\therefore (x+y)^3 - b^3 = -3a(x+y)(x+y-b),$$

$$\therefore (x+y-b)\{(x+y)^2 + b(x+y) + b^2 + 3a(x+y)\} = 0.$$

$\therefore$ either $x + y = b$, in which case $xy = 0$, $\therefore x = 0, b$; $y = b, 0$.

or $$(x+y)^2 + (3a+b)(x+y) + b^2 = 0.$$

This gives the value of $x + y$, and $xy$ being then known, the solution can be easily completed.

(2) $\qquad x + y + z = 3, \quad \therefore z = 3 - (x + y).$

$$3 = x^2 + y^2 + z^2 = x^2 + y^2 + 9 - 6(x + y) + (x + y)^2,$$

$$\therefore x^2 + y^2 + xy - 3(x + y) = -3.$$

$$6 = x^3 + y^3 + z^3 = x^3 + y^3 + 27 - 27(x + y) + 9(x + y)^2 - (x + y)^3,$$

$$\therefore 7 = 9(x + y) - 3(x + y)^2 + xy(x + y) \ldots (1)$$

and $\qquad 3 = 3(x + y) - (x + y)^2 + xy,$

$$\therefore 3(x + y) = 3(x + y)^2 - (x + y)^3 + xy(x + y) \ldots (2)$$

Subtract (2) from (1),

$$\therefore 7 = 12(x + y) - 6(x + y)^2 + (x + y)^3,$$

$$\therefore (x + y)^3 - 6(x + y)^2 + 12(x + y) - 8 = -1,$$

$$\therefore (x + y - 2)^3 = -1, \quad \therefore x + y - 2 = -1,$$

$$\therefore x + y = 1$$

and $\qquad xy = 1$

$$\therefore \quad \begin{cases} z = 2 \\ x = \frac{1}{2}\{1 \pm \sqrt{-3}\} \\ y = \frac{1}{2}\{1 \mp \sqrt{-3}\} \end{cases}$$

By symmetric changes of the letters, we can obtain two other sets of values for $x$, $y$ and $z$.

3. Since $\qquad (x + y)^2 = 4xy + (x - y)^2$

we see that if the product of two quantities is constant the sum is a min. when the quantities are equal. Now the product

$$\frac{2\cos\theta}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2\cos\theta} = 1$$

$\therefore$ the given exp. is a min. when

$$\frac{2\cos\theta}{\sqrt{3}} = \frac{\sqrt{3}}{2\cos\theta} \quad \therefore \cos\theta = \pm\frac{\sqrt{3}}{2} \quad \therefore \theta = 2n\pi \pm \frac{\pi}{6}.$$

4. The sides of the triangle formed by joining the centres of the circles are $b + c$, $c + a$, $a + b$. If $s$ be the semi-perimeter,

$$s = a + b + c, \; s - (b + c) = a, \; \&c.$$

$$\therefore r^2 = \frac{S^2}{s^2} = \frac{(a + b + c)\,abc}{(a + b + c)^2} = \frac{abc}{a + b + c}, \therefore \frac{1}{r^2} = \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}.$$

5. Tripos 1878. Monday morning. No. 6.

6. If $B$ be one extremity of the minor axis, $SB = AC$. $\therefore$ $A$ is the foot of the directrix of the parabola. $\therefore$ its vertex bisects $SA$. If we put $SB = CA'$ we see that another parabola can be described whose vertex bisects $SA'$.

7. Let $D$ be the point of suspension, $ACEB$ the rod, $E$ the position of the ring, and $C$ the point where the vertical through $D$ cuts the rod. Then $C$ is the middle point of the rod, and $DE$ is at right angles to $AB$. Draw $AF$ perpendicular to $DC$ produced. Let $DAB = a$. Then $AC = 1$ ft. $AD + DE = 9$ ft.

Since the tension of the string is the same throughout, $CD$ bisects the angle $ADE$, and since $DEA$ and $DFA$ are right angles, a circle will go round $DEFA$.

$$\therefore FDA = FAC = \theta; \quad \text{and} \quad DAF = \frac{\pi}{2} - ADF,$$

$$\therefore a + \theta = \frac{\pi}{2} - \theta, \quad \therefore a = \frac{\pi}{2} - 2\theta.$$

Now $1 = AC = AF \sec\theta = AD\cos(a+\theta)\sec\theta = AD\tan\theta$,

$$\therefore 9\tan\theta = (AD + AE)\tan\theta = (AD + AD\sin a)\tan\theta$$

$$= AD(1 + \sin a)\tan\theta$$

$$= 1 + \sin a = 1 + \cos 2\theta = 1 + \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{2}{1 + \tan^2\theta}.$$

$$\therefore 9\tan^3\theta + 9\tan\theta = 2.$$

If for $\tan\theta$ we substitute $3^{-\frac{1}{3}} - 3^{-\frac{2}{3}}$, the left-hand side becomes

$$= 9(3^{-\frac{1}{3}} - 3^{-\frac{2}{3}})^3 + 9(3^{-\frac{1}{3}} - 3^{-\frac{2}{3}})$$

$$= 9(3^{-1} - 3.3^{-\frac{4}{3}} + 3.3^{-\frac{5}{3}} - 3^{-2} + 3^{-\frac{1}{3}} - 3^{-\frac{2}{3}})$$

$$= 9(3^{-1} - 3^{-\frac{1}{3}} + 3^{-\frac{2}{3}} - 3^{-2} + 3^{-\frac{1}{3}} - 3^{-\frac{2}{3}})$$

$$= 9(3^{-1} - 3^{-2}) = 9\left(\frac{1}{3} - \frac{1}{9}\right) = 3 - 1 = 2.$$

$\therefore 3^{-\frac{1}{3}} - 3^{-\frac{2}{3}}$ is a root of the equation.

## PAPER XXXI.

1. Tripos 1878. Wednesday morning. No. 3.

2. Tripos 1878. Monday afternoon. No. 6.

3. (1) The given expression

$= \cos C(a\sin B - b\sin A) + \cos A\,(b\sin C - c\sin B) + \cos B(c\sin A - a\sin C)$

$= 0.$

(2) Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = R$ suppose.

$\therefore a = R\sin A,\ b = R\sin B,\ c = R\sin C.$

$$\therefore \frac{a^2 - b^2}{\cos A + \cos B} = R^2\frac{\sin^2 A - \sin^2 B}{\cos A + \cos B} = R^2\frac{\sin(A+B)\sin(A-B)}{2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}}$$

$$= 2R^2\sin\frac{A+B}{2}\sin\frac{A-B}{2} = 2R^2\left(\sin^2\frac{A}{2} - \sin^2\frac{B}{2}\right), \&c.$$

$$\therefore \text{given exp.} = 2R^2\left(\sin^2\frac{A}{2} - \sin^2\frac{B}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} + \sin^2\frac{C}{2} - \sin^2\frac{A}{2}\right)$$

$= 0.$

*4. Let $P$ be the point within the regular polygon $ABC\ldots$ of $n$ sides whose centre is $O$. From $P$ and $O$ let fall the perpendiculars $PM_1$, $ON_1$ on the side $AB$, and draw $PR_1$ perpendicular to $ON_1$, and use a similar construction and notation for each of the other sides. Let $OP = \delta$.

Then $ON_1 = ON_2 = \ldots =$ radius of inscribed circle $= r$.

Also $R_1$, $R_2\ldots$ lie on the circumference of the circle on $OP$ as diameter, and $R_1OR_2 = R_2OR_3 = \ldots$, each being the supplement of an angle of the polygon. $\therefore R_1R_2 = R_2R_3 = \ldots$

$\therefore R_1R_2R_3\ldots$ is a regular polygon.

Now since $N_1N_2\ldots N_n$ is a regular polygon,

$\therefore \Sigma PN_1^2 = n\,(ON_1^2 + OP^2) = n\,(r^2 + \delta^2)$. See Casey, Bk. IV., Prop. IV.

Similarly $$\Sigma PR_1^2 = n\left(\frac{OP^2}{4} + \frac{OP^2}{4}\right) = \frac{n}{2}\,\delta^2.$$

But $$PM_1^2 + PR_1^2 = PN_1^2$$

$$\therefore\ \Sigma PM_1^2 = \Sigma PN_1^2 - \Sigma PR_1^2 = n\,(r^2 + \delta^2) - \frac{n}{2}\,\delta^2 = nr^2 + \frac{n}{2}\,\delta^2.$$

5. Tripos 1875. Monday morning. No. 3.

6. Tripos 1875. Monday morning. No. 8.

*7. The equation $(x-a)^2+(y-b)^2=(ax+\beta y+\gamma)^2$

$$=(a^2+\beta^2)\frac{(ax+\beta y+\gamma)^2}{a^2+\beta^2}$$

represents a conic whose focus is $(a, b)$ directrix $ax+\beta y+\gamma=0$, and eccentricity $=\sqrt{a^2+\beta^2}$.

Now the necessary and sufficient conditions that two conics should be identical in magnitude and form are (1) the eccentricities must be equal; (2) the perpendicular distance of the focus from the directrix must be the same.

$$\therefore\ a^2+\beta^2=a'^2+\beta'^2 \quad \ldots\ldots\ldots \quad (1)$$

$$\frac{(a\alpha+b\beta+\gamma)^2}{a^2+\beta^2}=\frac{(a'\alpha+b'\beta'+\gamma')^2}{a'^2+\beta'^2},$$

or
$$(a\alpha+b\beta+\gamma)^2=(a'\alpha+b'\beta'+\gamma')^2 \quad \ldots\ldots \quad (2)$$

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## Paper XXXII.

1. Tripos 1878. Wednesday morning. No. 4.

2. Tripos 1878. Monday afternoon. No. 5.

3. $$\frac{\left(1-\tan\frac{\alpha}{2}\right)\left(1-\tan\frac{\beta}{2}\right)}{\left(1+\tan\frac{\alpha}{2}\right)\left(1+\tan\frac{\beta}{2}\right)}=\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}\cdot\frac{\frac{1}{\sqrt{2}}\cos\frac{\beta}{2}-\frac{1}{\sqrt{2}}\sin\frac{\beta}{2}}{\frac{1}{\sqrt{2}}\cos\frac{\beta}{2}+\frac{1}{\sqrt{2}}\sin\frac{\beta}{2}}$$

$$=\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}\cdot\frac{\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}=\frac{2\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\sin\frac{\alpha}{2}}{2\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)\cos\frac{\alpha}{2}}$$

$$=\frac{\sin\alpha-2\sin^2\frac{\alpha}{2}}{\sin\alpha+2\cos^2\frac{\alpha}{2}}=\frac{\sin\alpha+\cos\alpha-1}{\sin\alpha+\cos\alpha+1}=\frac{\sin\alpha+\sin\beta-1}{\sin\alpha+\sin\beta+1}.$$

4. $BP=ax, \quad \therefore CP=a(1-x); \quad CQ=bx.$

$$\therefore PQ^2=PC^2+CQ^2-2PC.CQ\cos C$$

$$=a^2(1-x)^2+b^2x^2-2abx(1-x)\frac{a^2+b^2-c^2}{2ab}$$

$$=x^2(2a^2+2b^2-c^2)-x(3a^2-b^2+c^2)+a^2.$$

So $$QR^2=x^2(2b^2+2c^2-a^2)-x(3b^2-c^2+a^2)+b^2.$$

$$RP^2=x^2(2c^2+2a^2-b^2)-x(3c^2-a^2+b^2)+c^2.$$

$$\therefore PQ^2+QR^2+RP^2=3x^2(a^2+b^2+c^2)-3x(a^2+b^2+c^2)+a^2+b^2+c^2$$

$$=(3x^2-3x+1)(a^2+b^2+c^2)$$

$$=\tfrac{1}{4}(a^2+b^2+c^2)+3(x-\tfrac{1}{2})^2(a^2+b^2+c^2).$$

5. Tripos 1875. Monday morning. No. 6.

6. Let $P$, $Q$ be two points on the conic, and let $SP$, $SQ$ make angles $\phi$, $\phi + a$ with the axis of $x$, $a$ being constant. Let the tangents at $P$ and $Q$ intersect in $R$.

The equation to $RP$ is $\frac{l}{r} = e\cos\theta + \cos(\theta - \phi)$;

" " $RQ$ " $\frac{l}{r} = e\cos\theta + \cos(\theta - \phi - a)$.

$\therefore$ at $R$, $e\cos\theta + \cos\overline{\theta - \phi} = e\cos\theta + \cos\overline{\theta - \phi - a}$,

$$\therefore \theta - \phi = 2\pi - \theta + \phi + a, \quad \therefore \theta - \phi = \pi + \frac{a}{2};$$

$\therefore$ the locus of $R$ is the conic

$$\frac{l}{r} = e\cos\theta - \cos\frac{a}{2}.$$

This may also be solved by reciprocating the theorem, 'The envelope of the chord of contact of tangents to a circle which cut at a constant angle is a concentric circle.'

7. Let the plane of the paper represent the section of the cone made by a plane containing the axis. Let $BC$, $DE$ be the diameters of the base and top of the frustum. Produce $BD$, $CE$ to meet in $A$. Let the axis of the cone cut $BC$ in $F$, and $DE$ in $G$. Let $H$, $K$, $L$ be the C. of G. of the cone $ABC$, the cone $ADE$, and the frustum $BDEC$ respectively. Let $LF = x$, $AG = GF = a$, and the angle $BAC = 2a$.

Then $$HL = \frac{a}{2} - x, HK = \frac{3a}{4}.$$

Then volume $ADE = 3AG$ . circle $DE = 3a\pi a^2 \tan^2 a$;

$$\therefore \text{volume } BDEC = ABC - ADE$$
$$= 6a\pi 4a^2 \tan^2 a - 3a\pi a^2 \tan^2 a$$
$$= 21a^3 \pi \tan^2 a.$$

$$\therefore \frac{\frac{a}{2} - x}{\frac{3a}{4}} = \frac{3a^3 \pi \tan^2 a}{21a^3 \pi \tan^2 a} = \frac{1}{7}; \quad \therefore x = \frac{11}{28}a$$

$$\therefore LG : LF :: 17 : 11.$$

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## Paper XXXIII.

1. Tripos 1875. Monday afternoon. No. 6.

2. Let $S_n = 1 + 2(1-a) + 3(1-a)(1-2a) + \ldots$
$$+ n(1-a)\ldots(1-\overline{n-1}a) + \frac{1}{a}(1-a)\ldots(1-na),$$
$$\therefore S_{n+1} = 1 + 2(1-a) + 3(1-a)(1-2a) + \ldots$$
$$+ (n+1)(1-a)\ldots(1-na) + \frac{1}{a}(1-a)\ldots(1-\overline{n+1}a),$$
$$\therefore S_{n+1} - S_n = (n+1)(1-a)\ldots(1-na) + \frac{1}{a}(1-a)\ldots(1-na)\{1-(n+1)a-1\}$$
$$= 0. \quad \therefore S_1 = S_2 = S_3 = \ldots$$

And $$S_1 = 1 + \frac{1-a}{a} = \frac{1}{a} = a^{-1},$$

$\therefore a^{-1} = S_n =$ expression on the right.

3. $$n \sin(n-1)\theta = \frac{2n \sin(n-1)\theta (1 - \cos\theta)}{2(1-\cos\theta)}$$
$$= \frac{2n \sin(n-1)\theta - n \sin n\theta - n \sin(n-2)\theta}{2(1-\cos\theta)}.$$

So $$(n-1)\sin(n-2)\theta$$

$$= \frac{2(n-1)\sin(n-2)\theta - (n-1)\sin(n-1)\theta - (n-1)\sin(n-3)\theta}{2(1-\cos\theta)};$$

. . . . . .

$$4\sin 3\theta = \frac{8\sin 3\theta - 4\sin 4\theta - 4\sin 2\theta}{2(1-\cos\theta)};$$

$$3\sin 2\theta = \frac{6\sin 2\theta - 3\sin 3\theta - 3\sin\theta}{2(1-\cos\theta)};$$

$$2\sin\theta = \frac{4\sin\theta - 2\sin 2\theta - 2\sin 0}{2(1-\cos\theta)}.$$

$$\therefore 2\sin\theta + 3\sin 2\theta + \ldots + n\sin(n-1)\theta$$

$$= \frac{(n+1)\sin(n-1)\theta + \sin\theta - n\sin n\theta}{2(1-\cos\theta)}$$

4. Since $r = \frac{S}{s}$,

$$\therefore ABC = \tfrac{1}{2}r(a+b+c),\ BPC = \tfrac{1}{2}r_1(\beta+\gamma+a),\ CPA = \tfrac{1}{2}r_2(\gamma+a+b),$$
$$APB = \tfrac{1}{2}r_3(a+\beta+c).$$

Now $$BPC+CPA+APB = ABC;$$

$$\therefore r_1(\beta+\gamma+a)+r_2(\gamma+a+b)+r_3(a+\beta+c) = r(a+b+c);$$

$$\therefore (r_2+r_3)a+(r_3+r_1)\beta+(r_1+r_2)\gamma = (r-r_1)a+(r-r_2)b+(r-r_3)c.$$

5. Tripos 1875. Wednesday morning. No. 1.

6. $$SP \,.\, S'P = CD^2 = CP^2.$$

7. Let $\alpha, \beta$ be the eccentric angles of $E$ and $F$,

$$\therefore EF = a^2(\cos\alpha - \cos\beta)^2 + b^2(\sin\alpha - \sin\beta)^2$$
$$= 4\sin^2\tfrac{1}{2}(\alpha-\beta)\left\{a^2\sin^2\tfrac{1}{2}(\alpha+\beta) + b^2\cos^2\tfrac{1}{2}(\alpha+\beta)\right\}$$
$$= 4\,CP^2\sin^2\tfrac{1}{2}(\alpha-\beta).$$

$$\therefore \frac{EF^2}{2CP^2} = 2\sin^2\tfrac{1}{2}(\alpha-\beta) = 1-\cos(\alpha-\beta) = 1-\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$= 1 - \frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2}.$$

## Paper XXXIV.

1. Tripos 1878. Monday afternoon. No. 4.

2. Tripos 1875. Wednesday morning. No. 5.

3. Expression $= 2\left(\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}\right) = \frac{1}{2}\left\{\left(1+\cos\frac{\pi}{4}\right)^2+\left(1+\cos\frac{3\pi}{4}\right)^2\right\}$

$$= \frac{1}{2}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2+\left(1-\frac{1}{\sqrt{2}}\right)^2\right\} = \frac{1}{2}\left(2+\frac{2}{2}\right) = \frac{3}{2}.$$

4. Tripos 1875. Thursday morning. No. 2.

5. Let $ABCD$ be the quadrilateral, $O$ the centre of the circle, $E$, $F$, $G$, $H$, the feet of the perpendiculars from $O$ on $AB$, $BC$, $CD$, $DA$. These points are the middle points of the respective sides. Join $DB$.

Then $AHE$ is $\frac{1}{4}$ $ADB$, and $CGF$ is $\frac{1}{4}$ $CDB$.

$$\therefore AHE + CGF = \tfrac{1}{4}\, ABCD.$$

Similarly, by joining $AC$ we may shew that

$$BEF + DHG = \tfrac{1}{4}\, ABCD.$$

$\therefore$ the sum of these four triangles is $\frac{1}{2}$ $ABCD$. $\therefore$ the remainder, the quadrilateral $EFGH$ is also half $ABCD$.

6. Let $Tt$, $T't'$, the tangents at $P$ and $Q$ meet the axis major in $T$, $T'$. Then $PSC = SPT + STP$, and $QSC = SQT' + ST'Q$.

$\therefore$ subtracting, $QSP = SQT' - SPT$, since $STP = ST'Q$.

Similarly $QS'P = S'Qt' - S'Pt$.

And $SQT' = S'Qt'$, $SPT = S'Pt$. $\therefore$ $QSP = QS'P$.

7. If $(x_1y_1)$, $(x_2y_2)$ be the coordinates of the points where the chord of contact intersects the curve, we must substitute the value of

$$\frac{y}{b} \text{ from } \frac{xh}{a^2}+\frac{yk}{b^2} = 1 \text{ in the equation } \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$$

Thus
$$\frac{y}{b} = \frac{b}{k}\left(1-\frac{xh}{a^2}\right).$$

$$\therefore 1 = \frac{x^2}{a^2}+\frac{b^2}{k^2}\left(1-\frac{xh}{a^2}\right)^2$$

$$= \frac{x^2}{a^2}\left(1+\frac{b^2h^2}{a^2k^2}\right) - 2\frac{b^2h}{a^2k}x + \frac{b^2}{k^2}.$$

$$\therefore x_1 + x_2 = 2\,\frac{b^2h}{a^2k} \div \frac{1}{a^2}\left(\frac{a^2k^2 + b^2h^2}{a^2k^2}\right) = 2h \div \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)$$

$$x_1x_2 = \left(\frac{b^2}{k^2} - 1\right) \div \frac{1}{a^2}\left(\frac{a^2k^2 + b^2h^2}{a^2k^2}\right) = \left(a^2 - \frac{a^2k^2}{b^2}\right) \div \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)$$

The values of $y_1 + y_2$ and $y_1y_2$ may be written down by interchanging $a$ and $b$, $h$ and $k$.

$$\therefore c^2 = \tfrac{1}{4}\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}$$

$$= \tfrac{1}{4}\{(x_1 + x_2)^2 - 4x_1x_2 + (y_1 + y_2)^2 - 4y_1y_2\}$$

$$= \left\{h^2 + k^2 - (a^2 + b^2)\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right) + \left(\frac{a^2k^2}{b^2} + \frac{b^2h^2}{a^2}\right)\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)\right\}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^{-2}$$

$$= \left(\frac{a^2k^2}{b^2} + \frac{b^2h^2}{a^2}\right)\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right)\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^{-2}.$$

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## Paper XXXV.

1. The number of combinations of 15 things taken 3 at a time is $= \dfrac{15\,.\,14\,.\,13}{1\,.\,2\,.\,3} = 5\,.\,7\,.\,13.$

$\therefore$ the number of days on which the girls could walk out having at least *one* different companion each day is $\dfrac{5\,.\,7\,.\,13}{5} = 7\,.\,13.$

Now suppose that on the same day $a$ and $b$ walk in the same row. If there were no restriction, they could do this 13 times, viz. once with each of the other girls; and similarly for each couple. But by the conditions of the question, they are only to go once. $\therefore$ the number of days $= \dfrac{7\,.\,13}{13} = 7.$

The question may also be considered thus.

Suppose we consider $a$. She can only walk with 14 different girls, and she must always walk with 2, but not with either of the same 2 again, and she can therefore only walk with a couple of different girls for 7 days; and similarly for the others. The order for these 7 days is

| 1st day. | 2nd day. | 3rd day. | 4th day. | 5th day. | 6th day. | 7th day. |
|---|---|---|---|---|---|---|
| $a, b, c$ | $a, e, g$ | $c, d, i$ | $d, h, j$ | $e, i, k$ | $c, h, o$ | $j, n, a$ |
| $d, e, f$ | $b, f, h$ | $b, g, n$ | $a, i, m$ | $h, l, n$ | $d, l, a$ | $f, k, c$ |
| $g, h, i$ | $i, j, o$ | $k, a, h$ | $f, g, l$ | $j, c, g$ | $e, j, b$ | $o, g, d$ |
| $j, k, l$ | $l, m, c$ | $m, f, j$ | $k, o, b$ | $m, b, d$ | $g, k, m$ | $h, m, e$ |
| $m, n, o$ | $n, d, k$ | $o, e, l$ | $n, c, e$ | $o, a, f$ | $i, n, f$ | $l, b, i$ |

For further information respecting this interesting problem see the *Ladies' and Gentlemen's Diary* for 1862 and 1863, and the *Proceedings of the London Mathematical Society* for 1881.

2. Tripos 1878. Monday afternoon. No. 5.

3. Tripos 1875. Thursday morning. No. 2.

*4. Let $OR_1$, $OR_2$ .... be the straight lines drawn from $O$ parallel to the sides of a regular polygon of $n$ sides, and on them let fall the perpendiculars $PR_1$, $PR_2$ .... from the point $P$.

Then $R_1$, $R_2$ .... lie on the circumference of the circle on $OP$ as diameter, and $R_1OR_2 = R_2OR_3 = \ldots$. each being the supplement of an angle of the polygon. $\therefore R_1R_2 = R_2R_3 = \ldots$.

$\therefore R_1R_2R_3 \ldots$. is a regular polygon,

$\therefore$ by Casey Bk. IV. Prop. 4, Cor. 1

$$\Sigma PR^2 = 2n \cdot \left(\frac{OP}{2}\right)^2 = \frac{n}{2} OP^2.$$

This may also be deduced from XXXI. No. 4, by supposing the polygon, whilst retaining its regular form, to become indefinitely diminished. It may then be regarded as a point coincident with $O$, the centre of figure, and the sides all pass through $O$ dividing the angle $2\pi$ into $n$ equal angles.

$$\therefore \Sigma PM_1^2 = nr^2 + \frac{n}{2}\delta^2 = \frac{n}{2}\delta^2, \text{ since } r = 0.$$

5. Let the plane of the ellipse $APA'$ be perpendicular to the plane of the paper. Let $O$ be the vertex of one of the cones from which it can be cut. Let the cone be cut by the plane of the paper which is supposed to contain the axis. In the triangle $OAA'$ inscribe the circle $ESF$, and on the other side of $AA'$ draw the escribed circle $HS'G$, so that $O$, $E$, $A$, $G$, are in one straight line.

Then $A'O - AO = (HO - HA') - (OG - AG) = HO - OG + AG - HA'$

$= AS' - A'S' =$ constant.

$\therefore$ the locus of $O$ is a hyperbola which has $A$ and $A'$ for its foci, and $S$, $S'$ for its vertices.

6. Take the centre of the circle as origin, and let $ACH = \theta$.

Then the coordinates of $A$ are $(a, 0)$; $B\ (-a, 0)$; $H\ (a\cos\theta,\ a\sin\theta)$; $K\ (a\cos\overline{\theta + 2\alpha},\ a\sin\overline{\theta + 2\alpha})$.

Equation to $AH$ is $y = -\dfrac{a\sin\theta}{a(1-\cos\theta)}(x-a) = -(x-a)\cot\dfrac{\theta}{2}$,

$$\therefore \frac{y}{\cos\frac{\theta}{2}} = \frac{-(x-a)}{\sin\frac{\theta}{2}} \quad \ldots\ldots\ldots \quad (1)$$

For $BK$, $y = \dfrac{a\sin(\theta+2\alpha)}{a\{1+\cos(\theta+2\alpha)\}}(x+a) = (x+a)\tan\left(\dfrac{\theta}{2}+\alpha\right)$,

$$\therefore y\cos\left(\frac{\theta}{2}+\alpha\right) = (x+a)\sin\left(\frac{\theta}{2}+\alpha\right),$$

$$\therefore y\left(\cos\frac{\theta}{2}\cos\alpha - \sin\frac{\theta}{2}\sin\alpha\right) = (x+a)\left(\sin\frac{\theta}{2}\cos\alpha + \cos\frac{\theta}{2}\sin\alpha\right); \quad (2)$$

Eliminating $\dfrac{\theta}{2}$ between (1) and (2),

$$y\{y\cos\alpha + (x-a)\sin\alpha\} = (x+a)\{y\sin\alpha - (x-a)\cos\alpha\},$$

$$\therefore x^2 + y^2 - 2ay\tan\alpha = a^2,$$

$$\therefore x^2 + (y - a\tan\alpha)^2 = a^2(1+\tan^2\alpha) = a^2\sec^2\alpha.$$

7. Using the same figure and letters as in XXXII. No. 7, let us take the case where $L$, the C. of G. of the frustum is vertically above $E$, $CE$ being supposed in contact with the plane.

Then $$CE = CF\cos FCA + FL\cos FAC,$$

$$\therefore a\sec\alpha = 2a\tan\alpha\sin\alpha + \frac{11}{28}a\,.\,\cos\alpha,$$

$$\therefore 1 = 2\sin^2\alpha + \frac{11}{28}\cos^2\alpha, \quad \therefore \sin^2\alpha = \frac{17}{45}.$$

And we see that if the vertical angle of the cone be decreased the perpendicular from $L$ will fall within the base $LE$; $\therefore$ the frustum will not topple over if the vertical angle of the cone is less than

$$2\sin^{-1}\sqrt{\frac{17}{45}}.$$

## Paper XXXVI.

1. $$\frac{1}{7} = \cdot 142857142857\ldots.$$

$$\therefore \frac{100}{7} = 14\cdot 28571428\ldots.$$

$$\therefore \frac{2}{7} = \frac{100}{7} - 14 = \cdot 285714\ldots.$$

Again $$\frac{10}{7} = 1\cdot 428571\ldots.$$

$$\therefore \frac{3}{7} = \frac{10}{7} - 1 = \cdot 428571\ldots.$$

Similarly for $\frac{4}{7}, \frac{5}{7}, \frac{6}{7}$.

Now $$\frac{1}{7} = \cdot\dot{1}42857\dot{} = \frac{142857}{999999},$$

$$\therefore \frac{7 \times 142857}{999999} = \frac{7 \times 1}{7} = 1,$$

$\therefore$ 142857 when multiplied by 7 must give a series of nines.

From the way in which $\frac{2}{7}, \frac{3}{7}, \ldots$ are formed, it is obvious that no new integers are introduced, and that the cyclical order is not changed. See also XXVI. No. 1, and LIX. No. 1.

2. Let $n = 7m + p$, $\therefore n - 1 = 7m + p - 1$,

$$\therefore \frac{n(n-1)}{7} = \text{integer} + \frac{p(p-1)}{7}.$$

By trial we find $p = -3$, or 4 are the only ones which give an odd remainder, and these are practically the same. Let $n = 7m + 4$. We have to find the sum of all terms of the series whose general term is

$$(7m + 3)(7m + 4), \text{ or } 49m^2 + 49m + 12.$$

$\therefore$ By Art. 26 the series to $m$ terms

$$= 49\frac{m(m+1)(2m+1)}{6} + 49\frac{m(m+1)}{2} + 12m.$$

Now if we put $m = 0$, $n = 4$, $\therefore n(n-1) = 12$, which is the first term, $\therefore$ to include this value of $m$, we must add the constant 12.

$$\therefore \text{ the sum } = 49m(m+1)\left\{\frac{2m+1}{6}+\frac{1}{2}\right\}+12(m+1)$$

$$= 49m(m+1)\,.\,\frac{m+2}{3}+12(m+1), \text{ and } m=\frac{n-4}{7};$$

$$= \frac{n+3}{7}\left\{(n-4)\,.\,\frac{n+10}{3}+12\right\}$$

$$= \frac{1}{21}(n+3)(n^2+6n-4).$$

3. $$\tan\frac{\alpha+\beta}{2} = \frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} = \frac{\tan^3\frac{\beta}{2}+\tan\frac{\beta}{2}}{1-\tan^3\frac{\beta}{2}\tan\frac{\beta}{2}}$$

$$= \frac{\tan\frac{\beta}{2}\left(1+\tan^2\frac{\beta}{2}\right)}{1-\tan^4\frac{\beta}{2}} = \frac{\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}} = \frac{1}{2}\tan\beta = \tan\phi.$$

$$\therefore n\pi+\phi = \frac{\alpha+\beta}{2}. \quad \text{If } n=0,\ \alpha+\beta = 2\phi.$$

4. Let $ABC$ be a triangle, $D$, $E$, $F$ the feet of the perpendiculars. Then by geometry we know that $A$, $B$, $C$ are the centres of the escribed circles of $DEF$. Let $\rho$, $r'$ be the radii of the circum- and in-scribed circles of $DEF$, $r_1r_2r_3$ the radii of the escribed circles, centres $A$, $B$, $C$. Then

$$x^2=\rho^2+2\rho r_1;\ y^2=\rho^2+2\rho r_2;\ z^2=\rho^2+2\rho r_3;\ p^2=\rho^2-2\rho r';\ \rho=\frac{R}{2}.$$

Now $$r_1+r_2+r_3-r' = 4\rho, \quad \text{XXII, 4.}$$

$$\therefore x^2+y^2+z^2+p^2 = 4\rho^2+2\rho\,.\,4\rho = 12\rho^2 = 3R^2.$$

5. Since the angles at $A$ and $D$ are right angles, a circle will go round $ABDE$. $\therefore CE\,.\,CA = CD\,.\,CB = \frac{1}{2}BC^2$.

6. A varying geometrical quantity has a maximum or minimum value when it has the same value for two consecutive positions. Let $BAC$ be the maximum triangle, $BA'C$ a consecutive position, indefinitely near to the first. Then the triangle $BAC = BA'C$. $\therefore AA'$, which is ultimately the tangent at $A$, is parallel to $BC$.

Similarly it may be shewn that the tangents at $B$ and $C$ are parallel to $AC$, $AB$. $\therefore$ the maximum triangle is equilateral.

$\therefore$ area $= 3BOC$, where $O$ is the centre of the circle,

$$= \frac{3}{2} \cdot BO \cdot OC \sin 120^\circ = \frac{3\sqrt{3}}{4} R^2.$$

For all questions of this kind, see *Theory of Maximum and Minimum treated without the aid of Differential Calculus* by the present writer.

7. Let $PG$ be the normal, $Q$ any point in it. Draw the ordinates $PN$, $QM$. On $SP$ let fall the perpendiculars $GK$, $QL$, and draw $QL'$ perpendicular to $PN$. Let $ASP = a$, $ASQ = \theta$, so that $(r, \theta)$ are the coordinates of $Q$.

Then $QL : GK :: PQ : PG :: PL' : PN$,

$\therefore QL : PL' :: GK : PN :: SG : SP :: SA : AX$. $\therefore QL = e \cdot PL'$.

$\therefore r \cdot \sin(\theta - a) = e(PN - QM) = e(SP \sin a - r \sin \theta)$;

$$\therefore e \sin \theta + \sin(\theta - a) = \frac{SP \cdot e \sin a}{r}$$

$$= \frac{e}{r} \cdot \frac{e \sin a}{1 + e \cos a}.$$

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## Paper XXXVII.

1. At the first observation let $x$ and $y$ be the angles made with the vertical by the hour and minute hands respectively.

Then at the 1st observation it is $30 + y$ min. past 4,

$\therefore$ the hour hand has gone $\frac{30 + y}{12}$ min. divisions,

$$\therefore x = 10 - \frac{30 + y}{12}, \quad \therefore 12x + y = 90 \quad . \quad . \quad . \quad (A)$$

At the 2nd observation it is $30 - x$ min. past 7,

$\therefore$ the hour hand has gone $\frac{30 - x}{12}$ min. divisions,

$$\therefore y = 5 + \frac{30 - x}{12}, \quad \therefore 12y + x = 90 \quad . \quad . \quad . \quad (B)$$

From $A$ and $B$ by subtraction, we get $x = y$.

2. (1) $\sqrt{\frac{a}{b}(bx-a^2)} - \sqrt{\frac{b}{a}(ax-b^2)} = a-b$, and evidently

$$\frac{a}{b}(bx-a^2) - \frac{b}{a}(ax-b^2) \equiv (a-b)x - \left(\frac{a^3}{b} - \frac{b^3}{a}\right)$$

$$\equiv (a-b)\left(x - \frac{a^3+a^2b+ab^2+b^3}{ab}\right),$$

$\therefore$ by division, we get

$$\sqrt{\frac{a}{b}(bx-a^2)} + \sqrt{\frac{b}{a}(ax-b^2)} = x - \frac{a^3+a^2b+ab^2+b^3}{ab}$$

$$\therefore 2\sqrt{\frac{a}{b}(bx-a^2)} = x + a - b - \frac{a^3+a^2b+ab^2+b^3}{ab}$$

$$= x - \frac{a^3+2ab^2+b^3}{ab}$$

$$= \frac{1}{a}\left(ax - \frac{a^3}{b}\right) - \frac{2ab+b^2}{a};$$

$$\therefore \left(ax - \frac{a^3}{b}\right) - 2a\sqrt{ax - \frac{a^3}{b}} - (2ab+b^2) = 0.$$

$$\therefore \sqrt{ax - \frac{a^3}{b}} = a \pm \sqrt{a^2+2ab+b^2} = a \pm (a+b) = 2a+b \text{ or } -b.$$

If we square both sides, we at once get the values of $x$.

(2) $$x + y + \sqrt{x^2-y^2} = a \quad \ldots\ldots\ldots \quad (1)$$

$$y\sqrt{x^2-y^2} = 2b \quad \ldots\ldots\ldots \quad (2)$$

From (1), $$\sqrt{x^2-y^2} = a - x - y \quad \ldots\ldots\ldots \quad (3)$$

$$\therefore x^2 - y^2 = a^2 + x^2 + y^2 + 2xy - 2ax - 2ay,$$

$$\therefore 2y^2 + 2xy - 2ay = 2ax - a^2.$$

From (2) and (3), $$ay - xy - y^2 = 2b \quad \ldots\ldots\ldots \quad (4)$$

$$2ax - a^2 + 4b = 0, \quad \therefore x = \frac{a^2-4b}{2a}.$$

Substituting this value of $x$ in (4) we get

$$y^2 - \frac{a^2 + 4b}{2a} y + 2b = 0,$$

which gives the value of $y$.

3. $A + B + C = n\pi$, $\therefore \sin(A + B) = \pm \sin C$, $\cos(A + B) = \mp \cos C$,

$$\frac{\sin 2C}{3} = \frac{\sin 2A}{5} = \frac{\sin 2B}{4} = \frac{\sin 2A + \sin 2B}{9} = \frac{\sin 2A - \sin 2B}{1},$$

$$\left.\begin{aligned} \therefore \frac{2}{9}\sin(A + B)\cos(A - B) &= \frac{2}{3}\sin C\cos C \\ 2\sin(A - B)\cos(A + B) &= \frac{2}{3}\sin C\cos C \end{aligned}\right\},$$

$$\therefore \cos(A - B) = \pm 3\cos C,\ \sin(A - B) = \mp \frac{1}{3}\sin C,$$

$$\therefore 1 = 9\cos^2 C + \frac{1}{9}\sin^2 C,\ \ \therefore \sec^2 C = 9 + \frac{1}{9}\tan^2 C,$$

$$\therefore 8\tan^2 C = 72,\ \ \therefore \tan^2 C = 9,\ \ \therefore \tan C = \pm 3,$$

$$\therefore \sin 2C = \frac{2\tan C}{1 + \tan^2 C} = \frac{\pm 6}{1 + 9} = \pm \frac{3}{5},$$

$\therefore \sin 2A = \pm 1$, $\therefore 2A = n\pi \pm \frac{\pi}{2}$, $\therefore A = \frac{n\pi}{2} \pm \frac{\pi}{4}$, $\therefore \tan A = \pm 1$.

Again

$$\left.\begin{aligned} \frac{2\sin(A + C)\cos(A - C)}{8} &= \frac{2\sin B\cos B}{4} \\ \frac{2\sin(A - C)\cos(A + C)}{2} &= \frac{2\sin B\cos B}{4} \end{aligned}\right\} \begin{aligned} \sin(A + C) &= \pm \sin B \\ \cos(A + C) &= \mp \cos B \end{aligned}$$

$$\therefore \cos(A - C) = \pm 2\cos B,\ \sin(A - C) = \mp \frac{1}{2}\sin B.$$

$$\therefore 1 = 4\cos^2 B + \frac{1}{4}\sin^2 B,\ \ \therefore \sec^2 B = 4 + \frac{1}{4}\tan^2 B,$$

$$\therefore 3\tan^2 B = 12,\ \ \therefore \tan^2 B = 4,\ \ \therefore \tan B = \pm 2.$$

4. Denote the two expressions on the left by $X$ and $Y$.

By the ordinary formula, $X = \dfrac{\sin n\beta}{2\sin\beta}$.

In finding $Y$ we will consider (1) $n$ odd, (2) $n$ even.

(1) when $n$ is odd;

$$Y = \cos(n-1)\beta + \cos(n-1-2)\beta + \ldots + \cos\{n-1-(n-3)\}\beta$$

$$\text{to } \frac{n-3}{2} + 1 \text{ terms.}$$

$$= \cos\left\{n-1-\left(\frac{n-1}{2}-1\right)\right\}\beta \sin\frac{n-1}{2}\beta \operatorname{cosec}\beta;$$

$$= \cos\frac{n+1}{2}\beta \sin\frac{n-1}{2}\beta \operatorname{cosec}\beta = \frac{1}{2}(\sin n\beta - \sin\beta)\operatorname{cosec}\beta;$$

$$\therefore X - Y = \frac{1}{2} = \frac{1+(-1)^{n-1}}{4}, \; n \text{ being odd.}$$

(2) when $n$ is even;

$$Y = \cos(n-1)\beta + \cos(n-1-2)\beta + \ldots + \cos\{n-1-(n-2)\}\beta$$

$$\text{to } \frac{n-2}{2} + 1 \text{ terms.}$$

$$= \cos\left\{n-1-\left(\frac{n}{2}-1\right)\right\}\beta \sin\frac{n\beta}{2}\operatorname{cosec}\beta$$

$$= \cos\frac{n\beta}{2}\sin\frac{n\beta}{2}\operatorname{cosec}\beta = \frac{1}{2}\sin n\beta \operatorname{cosec}\beta;$$

$$\therefore X - Y = 0 = \frac{1+(-1)^{n-1}}{4}, \; n \text{ being even.}$$

5. Let $PS$, $QR$ intersect in $L$. By symmetry $L$ lies on the diameter through $A$. It also lies on the polar of $A$, which is a fixed straight line. $\therefore$ $L$ is a fixed point.

6. By the same method which was employed in XXXVI. No. 6, we can shew that when a triangle inscribed in an ellipse has its maximum value, the tangent at any vertex is parallel to the opposite side, and $\therefore$ the diameter which joins any vertex to the centre bisects the opposite side, being conjugate to it. $\therefore$ the centre is the intersection of the bisectors of the sides.

We know that corresponding areas in the ellipse and auxiliary circle are in the ratio of $b : a$. Now by XXXVI. No. 6, the area of the maximum triangle in the auxiliary circle $= \frac{3\sqrt{3}}{4}a^2$. $\therefore$ the area of the maximum triangle which can be inscribed in the ellipse is $\frac{3\sqrt{3}}{4}ab$

Obviously, as in the circle, an infinite number of such triangles can be inscribed, all having the same area.

7. Let the coordinates of $P$ be $(x'y')$.

Then the coordinates of $U$ are $\frac{x}{2}(x' - ae), \frac{y'}{2}$.

$\therefore$ Equation to $AU$ is $\frac{y}{\frac{1}{2}y'} = \frac{x - a}{\frac{1}{2}(x' - ae) - a}$.

Equation to $SP$ is $\frac{y}{y'} = \frac{x - ae}{x' - ae}$.

$\therefore$ to determine the coordinates of $Q$ we have

$$\left.\begin{aligned} y(x' - ae - 2a) &= y'(x - a) \\ y(x' - ae) &= y'(x - a) \end{aligned}\right\},$$

$$\therefore x' = \frac{x - ae^2}{1 - e}, \; y' = \frac{y}{1 - e},$$

$\therefore$ if the equation to the original hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the equation to the locus of $Q$ is $\frac{(x - ae)^2}{a^2(1 - e)^2} - \frac{y^2}{b^2(1 - e)^2} = 1$.

The lines parallel to the asymptotes are evidently $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, shewing that the two hyperbolas have parallel asymptotes, and are $\therefore$ similar. Also the transverse axis $= a(e - 1) = CS - CA = AS$.

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## Paper XXXVIII.

1. $E$, the last dealer, has to pay away to the others $\frac{32}{2} \times 4 = 64$ counters. And as the total is $32 \times 5 = 160$, $E$ has $160 - 64 = 96$, which gives the last table. The others can be obtained by working backwards.

| | When $A$ deals. | $B$ | $C$ | $D$ | $E$ | At the end |
|---|---|---|---|---|---|---|
| $A$ has | 81 | 2 | 4 | 8 | 16 | 32 |
| $B$ „ | 41 | 82 | 4 | 8 | 16 | 32 |
| $C$ „ | 21 | 42 | 84 | 8 | 16 | 32 |
| $D$ „ | 11 | 22 | 44 | 88 | 16 | 32 |
| $E$ „ | 6 | 12 | 24 | 48 | 96 | 32 |
| | 160 | 160 | 160 | 160 | 160 | 160 |

2. Consider two consecutive numbers $n^{\frac{1}{n}}$ and $(n+1)^{\frac{1}{n+1}}$. Raise each to the $n(n+1)^{\text{th}}$ power, and divide by $n^n$. This gives us $n$ and $\left(1+\frac{1}{n}\right)^n$, the latter of which is equal to 2 + a proper fraction. $\therefore$ if $n > 2$, $n > \left(1+\frac{1}{n}\right)^n$; $\therefore$ the quantities gradually decrease.

Now consider $\sqrt[3]{3}$ and $\sqrt[2]{2}$, or $3^2$ and $2^3$. We see that $\sqrt{3} > \sqrt{2}$, $\therefore$ $\sqrt[3]{3}$ is the greatest of the numbers.

3. Since the tangents are in A.P. and $A = 45°$,

$$\therefore 2\tan B = \tan A + \tan C = 1 + \tan C = 1 - \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= 1 - \frac{1 + \tan B}{1 - \tan B} = -\frac{2\tan B}{1 - \tan B},$$

$\therefore$ either $\tan B = 0$, which is inadmissible,

or $\tan B - 1 = 1$, $\therefore$ $\tan B = 2$, $\therefore$ $\tan C = 3$.

$$\therefore \sin B = \frac{2}{\sqrt{5}}, \sin C = \frac{3}{\sqrt{10}},$$

$$\therefore 3 = \text{area} = \frac{1}{2} b^2 \frac{\sin A \sin C}{\sin B}, \therefore b^2 = 2.3\,\frac{2}{\sqrt{5}}\sqrt{2}\,\frac{\sqrt{10}}{3} = 8, \therefore b = 2\sqrt{2},$$

$$a = \frac{b\sin A}{\sin B} = 2\sqrt{2}\,\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{5}}{2} = \sqrt{5},$$

$$c = \frac{b\sin C}{\sin B} = 2\sqrt{2}\,\frac{3}{\sqrt{10}}\cdot\frac{\sqrt{5}}{2} = 3$$

4. $a' = 2r\cos\frac{A}{2}$ &c.

$$\therefore \frac{a'b'c'}{abc} = \frac{8r^3}{abc}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 8r^2 \cdot \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} \cdot \frac{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{abc}$$

$$= \frac{r^2}{2}\cdot\frac{2\sin B}{b}\cdot\frac{2\sin C}{c} = \frac{r^2}{2R^2}.$$

5. Let *abcd* be the figure formed, so that *B*, *a*, *b*, *B'*, are in this order in a straight line, and similarly *DcdD'*. Then since *AC'* is equal and parallel to *CA'*, $\therefore$ *AA'* is parallel to *CC'*. Similarly *BB'* is parallel to *DD'*. $\therefore$ the figure *abcd* is a parallelogram.

Join *AC*. Then since *D'* is the middle point of *AB*, and *D'd* is parallel to *Ba*, $\therefore$ *Ad* = *da*. $\therefore$ *abcd* = 2*Acd* = *AdD*.

Similarly it may be shewn that *abcd* is equal to each of the triangles *AaB*, *BbC*, *CcD*. But these 5 figures make up *ABCD*.

$$\therefore abcd = \frac{1}{5} ABCD.$$

6. Bisect *AB* in *C*. Draw through *C* a straight line perpendicular to *AB*. The point *P* will always lie on this straight line. Now the angle *QPA* = *PAB*. $\therefore$ *QP* is parallel to *AB*, and $\therefore$ perpendicular to *PC*; and *AQ* = *QP*. $\therefore$ the locus of *Q* is a parabola, focus *A*, and directrix *CP*.

7. Take *O* as origin, *OC* the axis of $x$, and a perpendicular to *OC* as the axis of $y$. Let the coordinates of the fixed point *C* be $(h, 0)$.

The equation to any straight line *CPP'* is of the form

$$y = m(x - h) \quad . \ . \ . \quad (1), \qquad \therefore y - mx = -mh.$$

The equation to the conic, which passes through the origin and touches the axis of $x$ is of the form

$$ax^2 + by^2 + cxy + ey = 0 \quad . \ . \ . \ . \ . \ . \quad (2)$$

If we make (2) homogeneous by means of (1) we get

$$ax^2 + by^2 + cxy - ey \cdot \frac{y - mx}{mh} = 0 \quad . \ . \ . \ . \quad (3)$$

This, being a homogeneous equation of the 2nd degree denotes two straight lines passing through the origin *O*, and since it is satisfied by the coordinates of the points which satisfy (1) and (2), it is evident that (3) represents *OP*, *OP'*. (3) may be written

$$x^2mah + mxy(ch + e) + y^2(mbh - e) = 0.$$

If this splits up into $(x - m_1 y)(x - m_2 y) = 0$,

$$m_1 = \cot COP, \ m_2 = \cot COP'.$$

$$\therefore \cot COP + \cot COP' = m_1 + m_2 = -\frac{m(ch + e)}{mah} = -\frac{ch + e}{ah}$$

$$= \text{constant.}$$

## Paper XXXIX.

**1.** Let $x, y, z$ denote the number of direct routes from London to Cambridge, London to Oxford, and Cambridge to Oxford respectively. Then $11 = x + yz$; and $13 = y + zx$.

$\therefore$ By subtraction $(x - y)(z - 1) = 2$.

Since $x, y, z$ are integers, the only solutions are from

either $z - 1 = 2$, $x - y = 1$, $\therefore z = 3$, $x = \frac{7}{2}$, $y = \frac{5}{2}$, inadmissible,

or $z - 1 = 1$, $x - y = 2$, $\therefore z = 2$, $x = 5, y = 3$.

**2.** Consider $p$ quantities each equal to $x^{q-r}$, $q$ quantities each $= x^{r-p}$, and $r$ quantities each $= x^{p-q}$.

$$\text{Their A.M.} = \frac{px^{q-r} + qx^{r-p} + rx^{p-q}}{p + q + r}.$$

$$\text{Their G.M.} = x^{\frac{p(q-r)+q(r-p)+r(p-q)}{p+q+r}} = x^0 = 1.$$

Since the A.M. $>$ the G.M.

$$\therefore \frac{px^{q-r} + qx^{r-p} + rx^{p-q}}{p + q + r} > 1,$$

except when $x = 1$, when the A.M. $=$ the G.M. which is also the case when $p = q = r$.

**3.** Produce $OC$ to meet the fixed circle in $E$.

Then $OD \cdot OC = OP \cdot OP' = OA \cdot OE =$ constant. And $OC$ is constant. $\therefore$ $OD$ is constant. $\therefore$ $D$ is a fixed point.

Join $PA$, $PD$, $CP$, $CP'$, $EP'$. Then since $PDCP'$ is a quadrilateral in a circle, $\therefore$ the angle $OPD = OCP'$. Similarly $OPA = OEP'$.

And $OCP'$ at the centre $=$ twice $OEP'$ at the circumference. $\therefore$ $OPD =$ twice $OPA$.

**4.** Since $O_1BO_2$ and $O_1AO_2$ are right angles, a circle will go round $O_1BAO_2$, $\therefore$ $OA \cdot OO_1 = OB \cdot OO_2$. Similarly it may be shewn that each of them $= OC \cdot OO_3$.

Again, $OA \cdot OO_1 = \dfrac{r}{\sin\frac{A}{2}} \cdot \dfrac{OB}{\sin\frac{C}{2}} = \dfrac{r \cdot r}{\sin\frac{A}{2} \cdot \sin\frac{B}{2}\sin\frac{C}{2}}$

$$= \frac{r \cdot a \sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = r \cdot \frac{a}{\sin\frac{A}{2}\cos\frac{A}{2}} = 4r \cdot \frac{a}{2\sin A} = 4rR.$$

5. Let $P$ be the position of the point within the triangle $ABC$. Let the perpendiculars from $P$ on $BC$, $CA$, $AB$ be denoted by $x$, $y$, $z$ respectively. Then evidently

$$ax + by + cz = 2\Delta \quad \ldots\ldots\ldots \quad (1)$$

We have to find when the expression $x^2 + y^2 + z^2$ is a minimum subject to the condition (1).

Through $P$ draw a straight line $PD$ parallel to $AB$, and suppose that $z$ remains constant whilst $x$ and $y$ vary. In other words, suppose $P$ to move along the line $PD$. Then we have to find the point on this line for which $x^2 + y^2$ is a minimum subject to the condition that $ax + by$ is constant.

Now $(ax + by)^2 + (bx - ay)^2 = (a^2 + b^2)(x^2 + y^2)$,

$\therefore$ $x^2 + y^2$ is a minimum when $bx - ay = 0$, *i.e.* when $\dfrac{x}{a} = \dfrac{y}{b}$.

Similarly, by supposing $x$ to remain constant whilst $y$ and $z$ vary, we shall get $\dfrac{y}{b} = \dfrac{z}{c}$. $\therefore$ If $x$, $y$, and $z$ all vary, the expression $x^2 + y^2 + z^2$ has its minimum value when

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{ax + by + cz}{a^2 + b^2 + c^2} = \frac{2\Delta}{a^2 + b^2 + c^2}$$

If $u^2$ be the minimum value of $x^2 + y^2 + z^2$,

$$u^2 = \frac{4\Delta^2}{(a^2 + b^2 + c^2)^2}(a^2 + b^2 + c^2) = \frac{4\Delta^2}{a^2 + b^2 + c^2}.$$

6. Square and add. $\therefore$ $x^2 + y^2 = A^2 + B^2 + 4AB\sin\theta\cos\theta$.

Multiply. $\therefore$ $xy = AB + (A^2 + B^2)\sin\theta\cos\theta$,

Eliminate $\theta$. $\therefore$ $\dfrac{x^2 + y^2 - (A^2 + B^2)}{4AB} = \dfrac{xy - AB}{A^2 + B^2}$;

$$\therefore (A^2 + B^2)(x^2 + y^2) - 4ABxy = (A^2 - B^2)^2.$$

7. Let $LMN$ be an equilateral triangle. Let $M$ be on the axis of $x$, $N$ on the axis of $y$. Let $(x, y)$ be the coordinates of $L$, and let $NMO = \theta$.

Then $x = LN \sin LNy = 2a \sin(150° - \theta) = 2a \sin(\theta + 30°)$

$= a\sqrt{3} \sin\theta + a\cos\theta,$

$y = LM \sin LMx = 2a \sin(\theta + 60°)$

$= a\sqrt{3} \cos\theta + a \sin\theta,$

$\therefore$ in No. 6 writing $a\sqrt{3}$ for $A$, and $a$ for $B$, we have

$$4(x^2 + y^2) \pm 4xy\sqrt{3} = a^2.$$

We have the double sign because $L$ may be on either side of $MN$.

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## Paper XL.

1. Since the number is less than a million, its cube root has only two figures.

Now $(10p + q)^3 = 1000p^3 + 300p^2q + 30pq^2 + q^3.$

$\therefore$ $q$ can at once be determined by considering the units' digit of the number, and we find $p$ by taking the number whose cube is the greatest integer which is not greater than the number given by the three left-hand figures of the given number.

2. Let £$x$ denote the price per head of the 10 sheep.

$\therefore$ £$(x - \frac{1}{2})$ denotes the price per head of the 5 sheep.

Let $10y + z$ = total price of the 10 sheep,

$\therefore$ $10y + z = 10x$ and $10z + y = 5x - \frac{5}{2}$,

$\therefore$ $8y - 19z = 5$;

$\therefore$ $y = 19r + 3$, $z = 8r + 1$.

Now $y$ and $z$ must be positive integers less than 10. $\therefore$ the only admissible values are obtained by putting $r = 0$;

$\therefore$ $y = 3$; $z = 1$; $\therefore$ $10x = 30 + 1 = 31$; $\therefore$ $x =$ £$3\frac{1}{10}$ = £3 2$s$.;

$x - \frac{1}{2} =$ £2 12$s$.

3. $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\ldots.$

$$=\frac{\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\ldots.}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$=\frac{\cos\frac{C}{2}\sin\frac{A+B}{2}+\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}}{\frac{1}{2}\cos\frac{C}{2}\left(\cos\frac{A+B}{2}+\cos\frac{A-B}{2}\right)}$$

$$=2\cdot\frac{\cos^2\frac{C}{2}+\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{C}{2}\sin\frac{C}{2}+\cos\frac{A-B}{2}\sin\frac{A+B}{2}}$$

$$=2\cdot\frac{1+\sin\frac{C}{2}\left(\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{C}{2}\right)}{\frac{1}{2}\sin C+\frac{1}{2}(\sin A+\sin B)}$$

$$=4\cdot\frac{1+\sin\frac{C}{2}\left(\cos\frac{A}{2}\cos\frac{B}{2}-\cos\frac{A+B}{2}\right)}{\sin A+\sin B+\sin C}$$

$$=4\cdot\frac{1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin A+\sin B+\sin C}.$$

4. Let $FAE$, $FBD$ be the tangents at $A$ and $B$; $DCE$ the tangent at $C$. Take $O$ the centre, and draw $OGD$ cutting $BC$ in $G$. Similarly draw $OHE$, $OKF$. These lines are respectively perpendicular to $BC$, $CA$, $AB$. Then $AF = R\cot AFO$, $AE = R\cot AEH$, $CD = R\cot CDO$.

$\therefore$ perimeter of $DEF = 2R(\cot AFO + \cot AEH + \cot CDO)$

$= 2R(\tan FAB + \tan EAC + \tan DCB)$

$= 2R(\tan C + \tan B + \tan A)$

$= 2R\tan A\tan B\tan C$, since $A+B+C = 180°$.

Now $\frac{a}{2R} = \sin A = \frac{2S}{bc}$; $\therefore R = \frac{abc}{4S}$;

$\therefore$ perimeter of $DEF = \frac{abc}{2S} \tan A \tan B \tan C$.

5. Draw $DG$ parallel to $CEF$, meeting $AB$ in $G$.

Then $\frac{AE}{ED} = \frac{AF}{FG} = \frac{2AF}{FB}$, since $FB = 2FG$, $D$ being the middle point of $BC$.

6. Let $S$ be the common focus, $H$, $H'$ the 2nd foci of the variable and fixed ellipses. Let the ellipses touch at $P$, and on the tangent at $P$ let fall the perpendiculars $SY$, $HZ$, $H'Z'$. Let $(a, b)$, $(a', b')$ be the axes of the variable and fixed ellipses.

Then since $SP$ makes with $PY$ the same angle which $PH$ and $PH'$ do, $\therefore$ $PH'H$ is a straight line.

(1) $SP + PH' = 2a'$; $SP + PH = 2a$. $\therefore HH' = 2(a - a') = \text{const.}$

$\therefore$ the locus of $H$ is a circle.

(2) $SY \cdot H'Z' = b'^2$; $SY \cdot HZ = b^2$. $\therefore \frac{H'Z'}{HZ} = \text{const.}$ $\therefore \frac{H'P}{HP} = \text{const.}$

$\therefore \frac{H'H}{H'P} = \text{const.}$ $\therefore$ the locus of $H$ is an ellipse similar to the fixed ellipse.

7. Let $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ be the feet of the normals from the point $P(\alpha, \beta)$. The equation to the normal at $P$ is $\beta - y = -\frac{y}{p}(\alpha - x)$. (1)

Eliminating $x$ between (1) and $y^2 = 2px$ we get

$$y^2 + 2p(p - \alpha)y = 2p^2\beta.$$

$\therefore$ $y_1, y_2, y_3$ being the roots of this equation,

$$y_1 + y_2 + y_3 = 0,\ y_2y_3 + y_3y_1 + y_1y_2 = 2p(p - \alpha).$$

Square the first of these equations, and subtract twice the second, and we get $x_1 + x_2 + x_3 = -2(p - \alpha)$.

From (1), $\frac{y_1^2}{2px_1} \cdot p(\beta - y_1) = -y_1(\beta - x_1)$,

$\therefore y_1(\beta - y_1) = -2\alpha x_1 + 2x_1^2$,

and $y_1^2 = -2px_1 + 2y_1^2$,

$\therefore$ adding, $2(x_1^2 + y_1^2) - 2(p + \alpha)x_1 - \beta y_1 = 0$ . . . . . (2)

$$\left.\begin{aligned} \therefore\ x_1^2 + y_1^2 - (p+a)x_1 - \frac{\beta}{2}y_1 &= 0 \\ \text{Similarly } x_2^2 + y_2^2 - (p+a)x_2 - \frac{\beta}{2}y_2 &= 0 \\ \text{and } x_3^2 + y_3^2 - (p+a)x_3 - \frac{\beta}{2}y_3 &= 0 \end{aligned}\right\} \quad . \ . \ . \quad (4)$$

Now $k^2=(x_1-a)^2+(y_1-\beta)^2+(x_2-a)^2+(y_2-\beta)^2+(x_3-a)^2+(y_3-\beta)^2$,

$\therefore$ by means of (4) we have

$$k^2 = 3(a^2+\beta^2) - (a-p)(x_1+x_2+x_3) - \left(2\beta+\frac{\beta}{2}\right)(y_1+y_2+y_3),$$
$$= 3(a^2+\beta^2) - 2(a-p)^2,$$

$\therefore$ the required locus is the ellipse

$$x^2 + 3y^2 + 4px - 2p^2 = k^2.$$

If $P$ be a point on the line such that the sum of the squares on the three normals is a minimum, and $= k^2$, then the sum is also a minimum for a point $P'$ on the line indefinitely near to $P$. Now $P$ and $P'$ both lie on the ellipse $x^2 + 3y^2 + 4px - 2p^2 = k^2$. $\therefore$ we must give $k$ such a value that the ellipse may touch the given line. Now the general equation to the tangent at $(x'y')$ to the ellipse referred to its centre, which is at $(-2p, 0)$, is

$$y - y' = -\frac{b^2x'}{a^2y'}(x-x') = -\frac{x'}{3y'}(x-x'). \quad \therefore\ m = -\frac{1}{3}\frac{x'}{y'},$$

or the point of contact lies on the line $3my + x = 0$. $\therefore$ changing the origin back to the point $(2p, 0)$ we have $3my + x + 2p = 0$.

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## Paper XLI.

1. 5 men do ·6006 of the work in 2·12 hrs.

$\therefore$ 1 man does $\frac{17}{300}$ " " " 1 hr.

$\therefore$ 3 men do $\frac{51}{100}$ " " " 3 "

$\therefore$ 7 boys do $\frac{49}{100}$ " " " 3 "

$\therefore$ 6 " " $\frac{7}{50}$ " " " 1 "

There remains $\frac{1198}{3000}$ of the work for 6 boys to do. $\therefore$ they will take $\frac{1198}{3000} \times \frac{50}{7}$ hrs. $= 2$ hrs. $51\frac{1}{7}$ min.

2. The expression on the left

$$= x(1-x)^{-1} - x^3(1-x^3)^{-1} + x^5(1-x^5)^{-1} \ldots$$

$$\left.\begin{aligned} &= x + x^2 + x^3 + x^4 + \ldots \\ &- x^3 - x^6 - x^9 - x^{12} - \ldots \\ &+ x^5 + x^{10} + x^{15} + x^{20} + \ldots \\ &- \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\} \text{Add the columns vertically.}$$

$$= x(1 - x^2 + x^4 - x^6 + \ldots) + x^2(1 - x^4 + x^8 - x^{12} + \ldots) + x^3(1 - x^6 + x^{12} - \ldots) + \ldots$$

$$= \frac{x}{1+x^2} + \frac{x^2}{1+x^4} + \frac{x^3}{1+x^6} + \ldots$$

3. $2 \sin(\alpha + \theta) \sin(\beta + \phi) = 2 \sin(\beta + \theta) \sin(\alpha + \phi)$,

$$\therefore \cos(\alpha - \beta + \theta - \phi) - \cos(\alpha + \beta + \theta + \phi) = \cos(\alpha - \beta - \theta + \phi) - \cos(\alpha + \beta + \theta + \phi),$$

$$\therefore \cos(\alpha - \beta + \theta - \phi) - \cos(\alpha - \beta - \theta + \phi) = 0,$$

$$\therefore 2 \sin(\alpha - \beta) \sin(\theta - \phi) = 0,$$

$$\therefore \text{either } \alpha - \beta \text{ or } \theta - \phi = n\pi.$$

4. Produce $AB_1C_1$ to meet $BC$ in $D$, and let $\theta = \frac{2m\pi}{6m+1}$.

Then each successive triangle will have its side inclined at an angle $\theta$ to the corresponding side of the previously formed triangle. $\therefore$ the $n^{\text{th}}$ triangle will have each side inclined at an angle $n\theta$ to the corresponding side of the original triangle. $\therefore$ $X$ is the triangle which is first formed when $n\theta$ is a multiple of $2\pi$, *i.e.* it is the $(6m+1)^{\text{th}}$ triangle, since $(6m+1)\,\theta = 2m\pi$.

$$\therefore X = S_{p+q}.$$

Now $A_1B_1C_1 = B_1DC + BDC_1 - A_1BC$

$$= ADC - AB_1C + ABD - AC_1B - A_1BC$$

$$= ABC - 3A_1BC$$

$$= \frac{\sqrt{3}}{4} BC^2 - \frac{3}{2} . BA_1 . A_1C \sin\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{4} BC^2 - \frac{3}{2} BC^2 . \frac{\sin\theta \sin\left(\frac{\pi}{3} - \theta\right)}{\frac{\sqrt{3}}{2}}$$

$$= \frac{\sqrt{3}}{4} BC^2 \left\{1 - 4 \sin\theta \sin\left(\frac{\pi}{3} - \theta\right)\right\}$$

$$= \frac{\sqrt{3}}{2} BC^2 \cos\left(2\theta - \frac{\pi}{3}\right).$$

$$\therefore S_1 = S . \cos\left(2\theta - \frac{\pi}{3}\right).$$

Similarly $S_2 = S_1 \cos\left(2\theta - \frac{\pi}{3}\right) = S . \cos^2\left(2\theta - \frac{\pi}{3}\right)$, and so on.

$$\therefore S_p = S . \cos^p\left(2\theta - \frac{\pi}{3}\right), \quad S_q = S . \cos^q\left(2\theta - \frac{\pi}{3}\right),$$

$$\therefore S_p . S_q = S . S . \cos^{p+q}\left(2\theta - \frac{\pi}{3}\right) = S . S_{p+q} = S . X.$$

5. Let $EA$, $EB$ be the tangents to the given circle. With centre $E$, and radius $EA$ describe a circle. These two circles will cut orthogonally, and $AB$ will be their radical axis. $\therefore$ if $F$ be any point in $AB$, and $FD$, $FG$ tangents to the two circles, $FD = FG$. $\therefore$ the circle described with centre $F$ and radius $FD$, cuts the given circle orthogonally, and also cuts orthogonally the circle described with centre $E$.

6. Let $CD$ be the diameter conjugate to $CP$, and let the circle on $MN$ as diameter cut the normal at $P$ in $E$ and $E'$. Draw $PF$ perpendicular to $CD$.

Then $PE^2 = PE . PE' = MP . PN = CD^2$. $\therefore EE' = 2CD$.

And $CE^2 = CP^2 + PE^2 - 2EP . PF = CP^2 + CD^2 - 2CD . PF$

$$= AC^2 + BC^2 - 2AC . BC$$

$$= (AC - BC)^2. \quad \therefore CE = AC - BC.$$

And $CE^2 + CE'^2 = 2(CP^2 + PE^2) = 2(CP^2 + CD^2) = 2(AC^2 + BC^2)$

$$= (AC + BC)^2 + (AC - BC)^2.$$

$$\therefore CE' = AC + BC.$$

7. Let $ABC$ be the given triangle, $O_1$ the centre of the given ellipse. Through the three points $A$, $B$, $C$ describe an ellipse of the same eccentricity as $O_1$, and let $O_2$ be its centre, and let its axes be parallel to those of $O_1$. Join $O_2A$, and draw $O_1a$ parallel to $O_2A$, $ac$ parallel to $AC$, and $cb$ parallel to $CB$. Then the triangle $abc$ will have its sides parallel to those of $ABC$.

Now consider any triangle inscribed in a circle. Through each angular point draw a straight line so that the line and the opposite side shall be parallel to two diameters at right angles. Then these three straight lines will evidently pass through a point, viz. the orthocentre. $\therefore$ by projection, we see that if in the triangle $abc$ we draw through the point $a$ a line $ad$ such that $ad$ and $bc$ are parallel to conjugate diameters, and similarly draw $be$ and $cf$, then $ad$, $be$, $cf$ will be concurrent. Then if through $A$, $B$, $C$ we draw $AD$, $BE$, $CF$ respectively parallel to these, they will evidently be concurrent.

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## Paper XLII.

1. $$(1-x^2)^n = (1+2x+x^2-2x-2x^2)^n$$
$$= \{(1+x)^2 - 2x(1+x)\}^n$$
$$= (1+x)^{2n} - 2nx(1+x)^{2n-1} + \frac{2n(2n-1)}{\underline{|2}}x^2.(1+x)^{2n-2}\ldots$$

2. Consider the series $1^3 + 2^3 + 3^3 + \ldots + (3m)^3$.

$$\text{The A. M.} = \frac{\text{sum}}{3m} = \left\{\frac{3m\,(3m+1)}{2}\right\}^2 \cdot \frac{1}{3m} = \frac{3m\,(3m+1)^2}{4}.$$

$$\text{The G. M.} = (\text{product})^{\frac{1}{m}} = \{\underline{|3m}\}^{\frac{1}{m}}.$$

Now A. M. $>$ G. M. $\therefore \frac{3m}{4}(3m+1)^2 > \{\underline{|3m}\}^{\frac{1}{m}}$ See Appendix.

3. By hypothesis $\frac{A+B}{C}, \frac{B+C}{A}, \frac{C+A}{B}$ are in A. P.

$$\therefore \frac{180^\circ - C}{C}, \frac{180^\circ - A}{A}, \frac{180^\circ - B}{B} \text{ are in A. P.}$$

$$\therefore \frac{180^\circ}{C} - 1, \frac{180^\circ}{A} - 1, \frac{180^\circ}{B} - 1 \text{ are in A. P.}$$

$$\therefore \frac{1}{C}, \frac{1}{A}, \frac{1}{B} \text{ are in A. P.}$$

$$\therefore C, A, B \text{ are in H. P.}$$

Let $C = 60° + x$, $A = 60° + y$, if possible, where $x$ and $y$ are positive.

Then $B = 180° - 60° - x - 60° - y = 60 - x - y$.

Since $C, A, B$ are in H. P.

$$\therefore C : B :: C - A : A - B,$$

$$\therefore 60 + x : 60 - x - y :: x - y : x + 2y.$$

Now the 1st term is $>$ 2nd, and 3rd term is $<$ 4th, which is impossible.

4. $A$ is any point on the circumference, $D$ the centre. Let the chord $OQPR$ which is parallel to $AB$ meet $AC$ in $P$, and the circumference in $Q, R$. Then the angle $OPC = BAC = OBC$. $\therefore P$ is on the circle round $OBC$. But $D$ is also on this circle, since $OBD + OCD = 2$ right angles. $\therefore DPO = DCO =$ a right angle. $\therefore QR$ is bisected in $P$.

5. Let $PN, RQ$ be the ordinates at $P$ and $R$.

$$PN^2 = RN.NA = 4AS.AN, \therefore RN = 4AS.$$

$$\therefore PR^2 = RN.RA = 4AS.AR = QR^2.$$

6. Let $B$ be the fixed point in the axis. Draw $BR$ perpendicular to the tangent at $P$ meeting $SP$ in $Q$. Draw $PK$ perpendicular to the directrix. Join $SK$ meeting $PR$ in $Y$. Then $SY$ is perpendicular to $PY$. Since $BQ$ is parallel to $SK$, the triangles $BQS, PSK$ are similar. $\therefore SQ : SB :: SP : PK, \therefore SQ = SB =$ const.

7. Draw $CM, ON$ perpendicular to $PQ$.

The equation to $PQ$ is $\frac{xh}{a^2} + \frac{yk}{b^2} - 1 = 0$.

$$\therefore CM^2 = \frac{a^4b^4}{h^2b^4 + k^2a^4},$$

and $PQ^2 = 4.\frac{(b^2h^2 + a^2k^2 - a^2b^2)(b^4h^2 + a^4k^2)}{(b^2h^2 + a^2k^2)^2}$. XXXIV., 7.

$$\therefore \text{area } CPQ = \tfrac{1}{2} CM.PQ = \frac{a^2b^2(b^2h^2 + a^2k^2 - a^2b^2)^{\frac{1}{2}}}{b^2h^2 + b^2k^2}.$$

Again, $$ON^2 = \frac{(b^2h^2 + a^2k^2 - a^2b^2)^2}{(b^4h^2 + a^4k^2)^2},$$

$$\therefore \text{ area } OPQ = \tfrac{1}{2}\, ON.PQ = \frac{(b^2h^2 + a^2k^2 - a^2b^2)^{\frac{3}{2}}}{b^2h^2 + a^2k^2},$$

$$\therefore \text{ area } CQOP = CPQ + OPQ$$

$$= \frac{a^2h^2\,(b^2h^2 + a^2k^2 - a^2b^2)^{\frac{1}{2}}}{b^2h^2 + a^2k^2} + \frac{(b^2h^2 + a^2k^2 - a^2b^2)\,(b^2h^2 + a^2k^2 - a^2b^2)^{\frac{1}{2}}}{b^2h^2 + a^2k^2}$$

$$= (b^2h^2 + a^2k^2 - a^2b^2)^{\frac{1}{2}}.$$

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## Paper XLIII.

1. Let $a(by+cz-ax) = b(cz+ax-by) = c(ax+by-cz) = R$ suppose.

$$\therefore \frac{R}{a} = by+cz-ax\,;\ \frac{R}{b} = cz+ax-by\,;\ \frac{R}{c} = ax+by-cz.$$

Adding the 2nd and 3rd of these results we get

$$2ax = R\left(\frac{1}{b} + \frac{1}{c}\right) = R\,.\,\frac{b+c}{bc},\quad \therefore x = R\,.\,\frac{b+c}{abc}.$$

Similarly $$y = R\,.\,\frac{c+a}{abc},\ z = R\,.\,\frac{a+b}{abc}$$

$$\therefore x + y + z = \frac{2R}{abc}(a + b + c) = 0.$$

2. $$(n+1)^3 = n\,(n-1)\,(n-2) + 6n\,(n-1) + 7n + 1,$$

$$\therefore \text{ the } n^{\text{th}} \text{ term of the series} = \frac{1}{\lfloor n-3} + \frac{6}{\lfloor n-2} + \frac{7}{\lfloor n-1} + \frac{1}{\lfloor n},$$

$$\therefore \text{ sum to } \infty = e + 6e + 7e + e = 15e.$$

3. Using the formulæ of Todh. *Trig.* § 288

$$\cos 5\theta = 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta\,;$$

$$\sin 5\theta = \sin\theta(1 - 12\cos^2\theta + 16\cos^4\theta)$$

$$= \sin\theta(5 - 20\sin^2\theta + 16\sin^4\theta)\,;$$

$$\therefore \sin 5\theta - \cos 5\theta = 5(\sin\theta - \cos\theta) - 20(\sin^3\theta - \cos^3\theta) + 16(\sin^5\theta - \cos^5\theta)$$
$$= (\sin\theta - \cos\theta)(1 - 2\sin 2\theta - 4\sin^2 2\theta);$$
$$\sin 5\theta + \cos 5\theta = 5(\sin\theta + \cos\theta) - 20(\sin^3\theta + \cos^3\theta) + 16(\sin^5\theta + \cos^5\theta)$$
$$= (\sin\theta + \cos\theta)(1 + 2\sin 2\theta - 4\sin^2 2\theta);$$

and $\tan\left(\theta - \frac{\pi}{4}\right) = \frac{\tan\theta - 1}{\tan\theta + 1} = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$

$\therefore$ by division we obtain the required result.

4. $$CP = \frac{a \sin PAC}{\sin\beta} \quad \ldots\ldots\ldots \quad (1)$$

$$CP = \frac{b \sin PBC}{\sin\beta} = -\frac{b \sin(PAC + \alpha + 2\beta)}{\sin\beta} \quad \ldots \quad (2)$$

$$\therefore a \sin PAC = b\{\sin PAC \cos(\alpha + 2\beta) + \cos PAC \sin(\alpha + 2\beta)\};$$

$$\therefore \tan PAC = -\frac{b \sin(\alpha + 2\beta)}{a + b\cos(\alpha + 2\beta)};$$

$$\therefore \sin PAC = \pm \frac{b \sin(\alpha + 2\beta)}{\sqrt{a^2 + b^2 + 2ab\cos(\alpha + 2\beta)}}.$$

By substituting this value of $\sin PAC$ in (1) we obtain the value of $CP$.

5. (1) Tripos 1875. Monday morning. No. 4.

(2) Let $CE$ be the common chord of the two circles.

Then the angle $CBD = BDC = CEA = ACE$.

$\therefore ABC = AEC$, and $ACB = ACE$, $\therefore DAC = EAC$.

$\therefore CE$ subtends at the circumference an angle which is one-fifth of two right angles. $\therefore CE$ subtends at the centre an angle which is one-fifth of four right angles. $\therefore CE$ is one-fifth of the circumference of the small circle.

6. Tripos 1875. Monday morning. No. 7.

7. This question is partly solved in V., No. 6. There we see that two parabolas can be described, and that the equations to the directrices are $x\cos\phi \pm y\sin\phi = a$. The lat. rect. are evidently twice the distances of $P(a\cos\phi, b\sin\phi)$ from these lines, and

$$= 2\{a - (a\cos^2\phi \pm b\sin^2\phi)\} = 2(a \mp b)\sin^2\phi.$$

## PAPER XLIV.

1. Tripos 1875. Monday afternoon. No. 4.

2. Let $$ax + b + \frac{c}{x} = y,$$

$$\therefore ax^2 - (y - b)x + c = 0,$$

$$\therefore x = \frac{y - b \pm \sqrt{(y-b)^2 - 4ac}}{2a}.$$

Since $x$ is real, the expression under the radical cannot be negative.

$$\therefore (y - b)^2 \not< 4ac, \quad \therefore y \not< b \pm 2\sqrt{ac}.$$

Again, let $$\frac{Ax^2 + Bx + C}{A'x^2 + B'x + C'} = y,$$

$$\therefore (A - A'y)x^2 + (B - B'y)x + C - C'y = 0,$$

$$\therefore 2(A - A'y)x = -(B - B'y) \pm \sqrt{(B - B'y)^2 - 4(A - A'y)(C - C'y)}.$$

Since $x$ is real, the expression under the radical cannot be negative, and $y$ will have its limiting values when this expression $= 0$. When this is the case, $2x(A - A'y) = -(B - B'y)$,

$$\therefore 2Ax + B = (2A'x + B')y = (2A'x + B') \cdot \frac{Ax^2 + Bx + C}{A'x^2 + B'x + C'}$$

$$\therefore (AB' - A'B)x^2 - 2(CA' - C'A)x + BC' - B'C = 0.$$

3. $2\sin A = \sin C + \cos C,$

$$\therefore \cos 2A = 1 - 2\sin^2 A = 1 - \tfrac{1}{2}(1 + 2\sin C\cos C) = \tfrac{1}{2}(1 - 2\sin C\cos C),$$

$$= \tfrac{1}{2}(\cos C - \sin C)^2 = \left(\frac{1}{\sqrt{2}}\cos C - \frac{1}{\sqrt{2}}\sin C\right)^2 = \cos^2\left(\frac{\pi}{4} + C\right),$$

$$\sin^2 B = \sin C\cos C$$

$$\therefore \tfrac{1}{2}\cos 2B = \tfrac{1}{2} - \sin^2 B = \tfrac{1}{2} - \sin C\cos C = \cos^2\left(\frac{\pi}{4} + C\right) = \cos 2A.$$

4. Tripos 1875. Monday morning. No. 5.

5. Tripos 1875. Monday morning. No. 11.

6. The equations to the two normals can evidently be written

$$y = m(x - 2a) - am^3 \quad \ldots\ldots\ldots \quad (1)$$

$$y = \frac{1}{m}(x - 2a) - \frac{a}{m^3} \quad \ldots\ldots\ldots \quad (2)$$

$$\therefore \text{ subtracting, } \frac{m^2 - 1}{m}(x - 2a) = a \cdot \frac{m^6 - 1}{m^3},$$

$$\therefore x - a = a \cdot \frac{(m^2 + 1)^2}{m^2} \quad \ldots\ldots\ldots \quad (3)$$

Again, eliminating $x - 2a$ from (1) and (2),

$$y(1 - m^2) = a\left(\frac{1}{m} - m^3\right), \quad \therefore y = a \cdot \frac{m^2 + 1}{m} \quad \ldots \quad (4)$$

$$\therefore \text{ from (3) and (4), } y^2 = a(x - a).$$

7. Let the tangent at $P$ meet the major axis in $R$, and the minor axis in $Q$, and let $(a\cos\phi, b\sin\phi)$ be the coordinates of $P$.

The equation to the tangent at $P$ is $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$,

$$\therefore CR = a\sec\phi, \quad CQ = b\operatorname{cosec}\phi,$$

$$\therefore QR^2 = CR^2 + CQ^2 = a^2\sec^2\phi + b^2\operatorname{cosec}^2\phi$$

$$= a^2 + b^2 + a^2\tan^2\phi + b^2\cot^2\phi,$$

$\therefore QR^2$ will be a min. when $a^2\tan^2\phi + b^2\cot^2\phi$ is a min.

Let $$a^2\tan^2\phi + b^2\cot^2\phi = u^2,$$

$$\therefore a^2\tan^4\phi - u^2\tan^2\phi + b^2 = 0,$$

$$\therefore \tan^2\phi = \frac{u^2 \pm \sqrt{u^4 - 4a^2b^2}}{2a^2}.$$

The min. value is given by $u^4 = 4a^2b^2$, $\therefore u^2 = 2ab$. Also see XLV. 2.

$$\therefore QR^2 = a^2 + b^2 + 2ab = (a + b)^2, \quad \therefore QR = a + b.$$

The singularity is evidently a min. for there is no max. limit.

*We add a purely geometrical proof of the above.

Let the tangent at $P$ meet the major and minor axes in $T$ and $t$. Then when $Tt$ is a min. $Tt = T't'$, where $T't'$ is the consecutive position of $Tt$, *i.e.* $Tt$ moves at this stage as if it were a line of constant length

sliding between two rectangular axes. Draw $TR$, $tR$ parallel to the axes meeting in $R$, the instantaneous centre. Then if $RP$ be joined, $RP$ is the normal at $P$.

$$\therefore PR^2 = PT \,.\, Pt = CD^2.$$

$\therefore PR = CD$, and is measured outwards along the normal.

$$\therefore CR = a + b. \quad \text{See XLI., 6.}$$

Since $CTRt$ is a rectangle, $Tt = CR = a + b$.

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## Paper XLV.

1. Let $S = 1 + 2x + 3x^2 + \ldots + (n-1)\,x^{n-2} + nx^{n-1}$,

$$\therefore Sx = \quad x + 2x^2 + \ldots + (n-1)\,x^{n-1} + nx^n,$$

$$\therefore S(1-x) = 1 + x + x^2 + \ldots\ldots + x^{n-1} - nx^n$$

$$= \frac{1-x^n}{1-x} - nx^n = \frac{1-(n+1)\,x^n + nx^{n+1}}{1-x},$$

$$\therefore S = \frac{1-(n+1)\,x^n + nx^{n+1}}{(1-x)^2}.$$

2. Since the product of the two expressions $ae^{kx}$ and $be^{-kx}$ is const. and $= ab$, $\therefore$ the sum is a min. when the two expressions are equal. Each of them is then $= \sqrt{\text{product}} = \sqrt{ab}$. $\therefore$ the sum $= 2\sqrt{ab}$.

3. (1) Let $S = \cos\theta \,.\, \sin\theta + \cos^2\theta \,.\, \sin 2\theta + \cos^3\theta \,.\, \sin 3\theta + \ldots$

$$C = \cos\theta \,.\, \cos\theta + \cos^2\theta \,.\, \cos 2\theta + \cos^3\theta \,.\, \cos 3\theta + \ldots$$

$$\therefore C + Si = \cos\theta \,.\, e^{\theta i} + \cos^2\theta \,.\, e^{2\theta i} + \cos^3\theta \,.\, e^{3\theta i} + \ldots$$

$$= \frac{\cos\theta \,.\, e^{\theta i}}{1 - \cos\theta \,.\, e^{\theta i}} = \frac{\cos\theta\,(\cos\theta + i\sin\theta)}{1 - \cos\theta\,(\cos\theta + i\sin\theta)}$$

$$= \frac{\cos\theta\,(\cos\theta + i\sin\theta)}{\sin\theta\,(\sin\theta - i\cos\theta)} = i\cot\theta \,.\, \frac{\cos\theta + i\sin\theta}{i\sin\theta + \cos\theta},$$

$\therefore S = \cot\theta$. Evidently $C = 0$.

(2) The given series = the sum of the two series

$$\log_e 2 + \frac{(\log_e 2)^2}{\lfloor 2} + \ldots + 2\log_e 2 + \frac{(2\log_e 2)^2}{\lfloor 2} + \ldots$$

$$= e^{\log_e 2} - 1 + e^{2\log_e 2} - 1 = 2 - 1 + 2^2 - 1 = 4.$$

4. Let $C = \cos x \sin\left(x + \frac{n\pi}{2}\right) + n\cos\left(x + \frac{\pi}{2}\right)\sin\left\{x + \frac{(n-1)\pi}{2}\right\} + \ldots$

$$S = \sin x \cos\left(x + \frac{n\pi}{2}\right) + n\sin\left(x + \frac{\pi}{2}\right)\cos\left\{x + \frac{(n-1)\pi}{2}\right\} + \ldots$$

$$\therefore C + S = \sin\left(2x + \frac{n\pi}{2}\right) + n\sin\left(2x + \frac{n\pi}{2}\right) + \ldots$$

$$= \sin\left(2x + \frac{n\pi}{2}\right)\left\{1 + n + \frac{n \cdot \overline{n-1}}{\underline{|2}} + \ldots\right\}$$

$$= (1+1)^n \sin\left(2x + \frac{n\pi}{2}\right) = 2^n \sin\left(2x + \frac{n\pi}{2}\right),$$

$$C - S = \sin\frac{n\pi}{2} + n\sin\left(\frac{n\pi}{2} - \pi\right) + \ldots$$

$$= \sin\frac{n\pi}{2}\left\{1 - n + \frac{n \cdot \overline{n-1}}{\underline{|2}} \ldots\right\}$$

$$= \sin\frac{n\pi}{2}(1-1)^n = 0,$$

$$\therefore 2C = 2^n \sin\left(2x + \frac{n\pi}{2}\right); \quad \therefore C = 2^{n-1}\sin\left(2x + \frac{n\pi}{2}\right).$$

5. Using the results given in Todh. *Trig.* Miscel. Ex. Nos. **257, 268,** the area of the triangle formed by joining the points of contact of the inscribed circle is $\frac{rS}{2R}$.

The area of the triangle formed by joining the centres of the escribed circles is

$$\frac{abc}{2s}\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right)$$

$$= \frac{abc}{2s}\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}, \text{ since } A + B + C = \pi.$$

$\therefore$ the product of the two is

$$\frac{rS}{2R} \cdot \frac{abc}{2s} \cdot \sqrt{\frac{s(s-a)}{(s-b)(s-c)} \cdot \frac{s(s-b)}{(s-c)(s-a)} \cdot \frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{rS}{2R} \cdot \frac{abc}{2s} \cdot \frac{\sqrt{s^4}}{\sqrt{s(s-a)(s-b)(s-c)}} \text{ and } abc = 4RS,$$

$$= S^2.$$

$\therefore$ $S$ is a mean proportional between the areas of the two triangles. The other cases in which one of the triangles is formed by joining the points of contact of one of the escribed circles can be proved in exactly the same manner.

6. Since $FD$ bisects the angle $BFC$,

$$\therefore\ DB : DC :: BF : FC.$$

Similarly $$CE : EA :: CF : FA,$$

$$\therefore\ CE . DB : CD . EA :: BF : FA,$$

$$\therefore\ CE . BD . AF = CD . BF . AE.$$

$\therefore$ the lines $AD$, $BE$, $CF$ are concurrent.

7. Let $ASP = \phi$, $ASQ = \phi + a$. Let the tangents at $P$ and $Q$ intersect in $T$.

The equation to $PT$ is $\frac{l}{r} = e\cos\theta + \cos(\theta - \phi)$,

" " $QT$ " $\frac{l}{r} = e\cos\theta + \cos(\theta - \phi - a)$,

$\therefore$ at $T$, $e\cos\theta + \cos(\theta - \phi) = e\cos\theta + \cos(\theta - \phi - a)$,

$\therefore \cos(\theta - \phi) = \cos(\theta - \phi - a)$, $\therefore\ \theta - \phi = \pm(\theta - \phi - a)$.

Taking the negative sign, $\theta = \phi + \frac{a}{2}$,

$$\therefore \text{ the locus of } T \text{ is } \frac{l}{r} = \cos\frac{a}{2} + e\cos\left(\phi + \frac{a}{2}\right),$$

$$\text{or } \frac{l\sec\frac{a}{2}}{r} = 1 + e\sec\frac{a}{2}\cos\left(\phi + \frac{a}{2}\right).$$

This is the equation to a conic, whose axis is inclined to the former at an angle $\frac{a}{2}$, and whose eccentricity is $e\sec\frac{a}{2}$.

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## Paper XLVI.

1. Put $x = a + \frac{1}{b} + \frac{1}{x}$, $\therefore\ bx^2 - abx - a = 0$. . . . . (1)

If $x = b + \frac{1}{c} + \frac{1}{x}$, $\therefore\ cx^2 - bcx - b = 0$. . . . . (2)

If $x = c + \frac{1}{a} + \frac{1}{x}$, $\therefore\ ax^2 - acx - c = 0$. . . . . (3)

Multiplying (1), (2), (3) together, and dividing by $abc$, we get

$$x^6-(a+b+c)x^5-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-bc-ca-ab\right)x^4$$

$$+\left(a+b+c-abc+\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)x^3$$

$$+\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}-bc-ca-ab\right)x^2-(a+b+c)x-1=0,$$

or $x^6-1-(a+b+c)x(x^4+1)+(ab+bc+ca)x^2(x^2-1)+(a+b+c-abc)x^3$

$$=x^4\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)-x^3\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)-x^2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right),$$

$\therefore$ the three continued fractions satisfy the given equation.

From (1) $$x=\frac{ab\pm\sqrt{a^2b^2+4ab}}{2b}.$$

The positive sign gives us $a+\frac{1}{b}+\frac{1}{a}+\cdots$

The negative sign gives us $-\frac{1}{b}+\frac{1}{a}+\frac{1}{b}+\cdots$

From (2) and (3) we obtain similar values.

$\therefore$ the other roots of the given equation are

$$-\frac{1}{b}+\frac{1}{a}+\ldots,\quad -\frac{1}{c}+\frac{1}{b}+\ldots,\quad -\frac{1}{a}+\frac{1}{c}+\ldots.$$

2. $$\left(\frac{4}{3}\right)^2\div\frac{8}{9}=2,$$

$$\therefore \log_e 2=\log\left(\frac{4}{3}\right)^2-\log\frac{8}{9}=2\log\left(1+\frac{1}{3}\right)-\log\left(1-\frac{1}{3^2}\right)$$

$$=2\left(\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1}{3}\cdot\frac{1}{3^3}-\ldots\right)+\left(\frac{1}{3^2}+\frac{1}{2}\cdot\frac{1}{3^4}+\frac{1}{3}\cdot\frac{1}{3^6}+\ldots\right).$$

3. The common denominator of the three fractions being $-\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)$, the numerator is

$$\sin(\theta-\beta)\sin(\theta-\gamma)\sin(\beta-\gamma)+\sin(\theta-\gamma)\sin(\theta-\alpha)\sin(\gamma-\alpha)$$
$$+\sin(\theta-\alpha)\sin(\theta-\beta)\sin(\alpha-\beta),$$

$$= \tfrac{1}{2}\sin(\beta-\gamma)\{\cos(\beta-\gamma)-\cos(2\theta-\beta-\gamma)\}$$
$$+ \tfrac{1}{2}\sin(\gamma-\alpha)\{\cos(\gamma-\alpha)-\cos(2\theta-\gamma-\alpha)\}$$
$$+ \tfrac{1}{2}\sin(\alpha-\beta)\{\cos(\alpha-\beta)-\cos(2\theta-\alpha-\beta)\}$$
$$= \tfrac{1}{4}\{\sin 2(\beta-\gamma)+\sin 2(\theta-\beta)-\sin 2(\theta-\gamma)\}$$
$$+ \tfrac{1}{4}\{\sin 2(\gamma-\alpha)+\sin 2(\theta-\gamma)-\sin 2(\theta-\alpha)\}$$
$$+ \tfrac{1}{4}\{\sin 2(\alpha-\beta)+\sin 2(\theta-\alpha)-\sin 2(\theta-\beta)\}$$
$$= \tfrac{1}{4}\{\sin 2(\alpha-\beta)+\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)\}$$
$$= \tfrac{1}{2}\{\sin(\alpha-\beta)\cos(\alpha-\beta)+\sin(\beta-\alpha)\cos(\alpha+\beta-2\gamma)\}$$
$$= \tfrac{1}{2}\sin(\alpha-\beta)\{\cos(\alpha-\beta)-\cos(\alpha+\beta-2\gamma)\}$$
$$= \sin(\alpha-\beta)\sin(\alpha-\gamma)\sin(\beta-\gamma)$$
$$= -\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha).$$

$\therefore$ given expression $= 1$.

4. Let $B$, $C$ denote the bases of the hill and tower respectively. $A$ and $D$ their summits. Then $ACB = \alpha$, $ADB = \beta$, $CD = a$.

Let $$AB = x,\ DAC = \theta.$$

Then $$CDB + \beta + \theta + \frac{\pi}{2} - \alpha = \pi.$$

$$\therefore\ CDB = \frac{\pi}{2} - (\beta + \theta - \alpha);\ CDA = \frac{\pi}{2} + \alpha - \theta,$$

$$\frac{a}{AC} = \frac{\sin\theta}{\cos(\alpha-\theta)};\ \frac{AC}{x} = \frac{1}{\sin\alpha},\ \therefore\ x\sin\theta = a\sin\alpha\cos(\alpha-\theta),$$

$$\therefore\ \sin\theta(x - a\sin^2\alpha) = a\sin\alpha\cos\alpha\cos\theta,$$

$$\therefore\ \frac{a\sin\alpha\cos\alpha}{\sin\theta} = \frac{x - a\sin^2\alpha}{\cos\theta} \quad \ldots\ldots\ (1)$$

Now $CB = x\cot\alpha$, $DB = a\operatorname{cosec}(\beta+\theta-\alpha)$, $CD^2 + CB^2 = DB^2$,

$$\therefore\ a^2 + x^2\cot^2\alpha = a^2\{1 + \cot^2(\beta+\theta-\alpha)\},$$

$$\therefore\ x\cot\alpha = a\cot(\beta+\theta-\alpha)$$

$$\therefore\ x\cos\alpha\sin(\beta+\theta-\alpha) = a\sin\alpha\cos(\beta+\theta-\alpha),$$

$$\therefore\ \sin\theta\{x\cos\alpha\cos(\beta-\alpha) + a\sin\alpha\sin(\beta-\alpha)\}$$
$$= \cos\theta\{a\sin\alpha\cos(\beta-\alpha) - x\cos\alpha\sin(\beta-\alpha)\}.$$

Eliminate $\theta$ by means of (1), and arrange the result in descending powers of $x$. We have

$$x^2\cos\alpha\sin(\beta-\alpha)+\alpha x\sin\alpha\left\{\cos^2\alpha\cos(\beta-\alpha)-\cos(\beta-\alpha)-\sin\alpha\cos\alpha\sin(\beta-\alpha)\right\}$$
$$+\alpha^2\sin^2\alpha\left\{\cos\alpha\sin(\beta-\alpha)+\sin\alpha\cos(\beta-\alpha)\right\}=0.$$

$$\therefore\ x^2\cos\alpha\sin(\beta-\alpha)-\alpha x\sin^2\alpha\sin\beta+\alpha^2\sin^2\alpha\sin\beta=0.$$

This equation is unaltered by writing $\pi-\alpha$, $\pi-\beta$ for $\alpha$, $\beta$. Now $\beta$ may be $>\frac{\pi}{2}$, but $\alpha$ must be $<\frac{\pi}{2}$. $\therefore$ only one value of $x$ satisfies the given conditions. The other value gives the solution of some other question.

5. Let $ABCD$ be the square, $P$ the point on $AC$ through which the four circles pass. Let $EFGH$ be the quadrilateral, $E$, $F$, $G$, $H$ being points on the circles round $PAB$, $PBC$, $PCD$, $PDA$ respectively.

The angle $PGD = PCD = \frac{1}{2}$ right angle.

The angle $PHD = PAD = \frac{1}{2}$ right angle.

$$\therefore\ PGD = PHD,\ \therefore\ PG = PH.$$

Also $PD = PB$, $\therefore$ the circles round $APD$ and $APB$ are equal.

$\therefore$ the angle $PHA = PEA$, $\therefore\ PH = PE$.

Similarly $PFC = PGC$, $\therefore\ PF = PG$.

$\therefore$ the circle described with centre $P$ and radius any one of the lines $PE$, $PF$, $PG$, $PH$ will pass through the extremities of the other three.

Also $PHG = PGH = PCD = \frac{1}{2}$ right angle. $\therefore\ HPG$ is a right angle.

And $PEF = PFB = PCB = \frac{1}{2}$ right angle. $\therefore\ EPF$ is a right angle.

$\therefore\ HPG = EPF$. Add to each $FPG$. $\therefore\ EPG = FPH$, and the sides which include them are equal. $\therefore$ the base $EG$ = base $FH$.

Since the triangles $EPG$, $HPF$ are equal in all respects, and $PE$ is perpendicular to $PF$ and $PG$ to $PH$, $\therefore$ the base $EG$ is at right angles to $FH$.

6. Let $S$ be the given focus, $C$ the centre of the auxiliary circle. Produce $SC$ to $S'$ so that $CS' = CS$. Then $S'$ is the other focus, and if $SS'$ intersect the circle in $A$ and $A'$, these points are the vertices. In $CS$ take a point $X$ such that $CX : CA :: CA : CS$. Then $X$ is the foot of the directrix. $\therefore$ since we know the focus, directrix and eccentricity, we can describe the curve.

7. The equation to a chord $PP'$ is $\frac{l}{r} = e\cos\theta + \sec\beta\cos(\theta - a)$.

In the rectangular hyperbola, $e = \sqrt{2}$, and $\sec\beta = \sqrt{2}$, since $\frac{1}{2}PSP' = \frac{\pi}{4}$.

$\therefore$ equation to $PP'$ is $\frac{l}{r\sqrt{2}} = \cos\theta + \cos(\theta - a)$, which is the equation of the tangent to a confocal and coaxial parabola.

This may also be proved by reciprocating the theorem, 'The locus of the intersection of tangents to a circle which cut at right angles is a concentric circle,' the centre of reciprocation being taken on the circumference of the inner circle.

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## Paper XLVII.

1. $\sqrt{x^2 + x(b-c) + a^2} + \sqrt{x^2 + x(c-a) + b^2} = -\sqrt{x^2 + x(a-b) + c^2}$.

Square and transpose.

$$\therefore x^2 + 2x(b-a) + a^2 + b^2 - c^2 = -2\sqrt{\{x^2 + x(b-c) + a^2\}\{x^2 + x(c-a) + b^2\}},$$

$$\therefore x^4 + 4x^3(b-a) + 2x^2\{a^2 + b^2 - c^2 + 2(b-a)^2\} + 4x(b-a)(a^2 + b^2 - c^2)$$
$$+ a^4 + b^4 + c^4 + 2a^2b^2 - 2a^2c^2 - 2b^2c^2$$
$$= 4\left[x^4 + x^3(b-a) + x^2(a^2 + b^2 + bc - ba - c^2 + ac)\right.$$
$$\left. + x\{a^2(c-a) + b^2(b-c)\} + a^2b^2\right],$$

$$\therefore 3x^4 - 2x^2(a^2 + b^2 + c^2 - 2bc - 2ca - 2ab) + 4x(a-b)(ac + bc - ab - c^2)$$
$$- (a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2) = 0,$$

$$\therefore x^4 + 2px^2 + \tfrac{4}{3}x(a-b)(b-c)(c-a) + r = 0,$$

$$\therefore x^4 + 2px^2 + qx + r = 0,$$

where $p$, $q$, and $r$ satisfy the required conditions.

2. From the $r$ things there can be formed $\lfloor r$ permutations.

„ $n-r$ „ „ „ $\underline{n-r}$ „

$$\therefore f(r) = \lfloor r \, \underline{\lfloor n-r}.$$

$\therefore$ the expression on the left hand

$$= \frac{1}{\lfloor n} + \frac{1}{\lfloor n-1} + \frac{1}{\lfloor 2 \lfloor n-2} + \ldots + \frac{1}{\lfloor n-1} + \frac{1}{\lfloor n}$$

$$= \frac{1}{\lfloor n} \left\{1 + n + \frac{n \cdot n-1}{\lfloor 2} + \ldots + n + 1\right\}$$

$$= \frac{1}{\lfloor n} (1+1)^n = \frac{2^n}{\lfloor n}.$$

3. (1) $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$, $\therefore \cos^2 18 = 1 - \frac{6-2\sqrt{5}}{16} = \frac{10+2\sqrt{5}}{16}$,

$$\therefore \tan^2 18^\circ = \frac{6-2\sqrt{5}}{10+2\sqrt{5}}.$$

$$\tan 36^\circ = \frac{2\tan 18^\circ}{1-\tan^2 18^\circ} = \frac{2\tan 18^\circ}{1-\frac{3-\sqrt{5}}{5+\sqrt{5}}} = \frac{2(5+\sqrt{5})}{2+2\sqrt{5}} \tan 18^\circ = \sqrt{5}\tan 18^\circ.$$

(2) $\tan 9^\circ = \dfrac{\sin 18^\circ}{1+\cos 18^\circ} = \dfrac{\frac{\sqrt{5}-1}{4}}{1+\frac{\sqrt{10+2\sqrt{5}}}{4}} = \dfrac{\sqrt{5}-1}{4+\sqrt{10+2\sqrt{5}}}$

$$= \frac{\sqrt{5}-1}{16-(10+2\sqrt{5})} \left\{4 - \sqrt{10+2\sqrt{5}}\right\}.$$

And $\dfrac{\sqrt{5}-1}{6-2\sqrt{5}} = \dfrac{\sqrt{5}-1}{(\sqrt{5}-1)^2} = \dfrac{1}{\sqrt{5}-1} = \dfrac{\sqrt{5}+1}{4}$,

$$\therefore \tan 9^\circ = \frac{\sqrt{5}+1}{4} \left\{4 - \sqrt{10+2\sqrt{5}}\right\}.$$

4. Let $ABC$ be the triangle required. From the vertex $A$ draw $AD$ perpendicular to the base $BC$. Then we have given the values of $BC$, $AD$, and $AB^2 \sim AC^2$.

Then $$AB^2 \sim AC^2 = (BD^2 + AD^2) \sim (AD^2 + DC^2)$$
$$= BD^2 \sim DC^2 = BC\,(BD \sim DC).$$

$\therefore$ $BD \sim DC$ is known. And $BD + DC$ is known. $\therefore$ $BD$ and $DC$ are known. $\therefore$ the position of $D$ is known. And since we know the value of $DA$ we can construct the triangle.

5. Let the tangents $OQ$, $OP'$ meet the directrix in the points $F'$ $F$. Draw the tangents $FP$, $F'Q'$. Let $OQ$, $FP$ intersect in $A$, and $FP'$, $F'Q'$ in $B$. Then $SA$ bisects the angle $PSQ$, and $SB$ bisects the angle $P'SQ'$. Since $F$ and $F'$ are on the directrix, $\therefore$ $PP'$ and $QQ'$ pass through $S$. $\therefore$ $AB$ passes through $S$.

Again, since $OSQ = OSP'$, and $QSA = P'SB$, $\therefore$ $OSA = OSB$. $\therefore$ $OS$ is at right angles to $AB$.

This may also be treated as an instructive example in reciprocal polars.

Take $EF$, any chord of a circle, and draw the diameters $EG$, $FH$. Corresponding to the lines $EF$, $FG$, $EH$ we have the points $O$, $A$, $B$. Since $FG$ and $EH$ are parallel, and $\therefore$ meet at infinity, we see that $AB$ passes through $S$.

Since $FG$ is at right angles to $EF$, we see that $AO$ subtends a right angle at $S$.

6. Let $AB$ be the circular arc, $O$ its centre, and let the particles be situated at the points $A$, $P$, $Q$, ... $B$. Let $AOP = \theta$, so that $(n-1)\,\theta = 2\alpha$. Take $O$ as origin, $OA$ as axis of $x$, and a line through $O$ at right angles to $OA$ as axis of $y$. Then if $(x_1 y_1)$, $(x_2 y_2)$ ... be the coordinates of $A$, $P$, ... and if $(\bar{x}, \bar{y})$ be the coordinates of $G$,

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$$

$$= \frac{r}{n}(1 + \cos\theta + \cos 2\theta + \ldots + \cos\overline{n-1}\,\theta),$$

$$= \frac{r}{n}\cos\frac{n-1}{2}\theta \sin\frac{n\theta}{2}\operatorname{cosec}\frac{\theta}{2} = \frac{r}{n}\cos\alpha \sin\frac{n\alpha}{n-1}\operatorname{cosec}\frac{\alpha}{n-1},$$

$$\bar{y} = \frac{1}{n}(y_1 + y_2 + \ldots + y_n),$$

$$= \frac{r}{n}(\sin\theta + \sin 2\theta + \ldots + \sin\overline{n-1}\,\theta), \text{ since } y_1 = 0,$$

$$= \frac{r}{n}\sin\frac{n-1}{2}\theta \sin\frac{n\theta}{2}\operatorname{cosec}\frac{\theta}{2} = \frac{r}{n}\sin\alpha \sin\frac{n\alpha}{n-1}\operatorname{cosec}\frac{\alpha}{n-1},$$

$$\therefore OG = \sqrt{\bar{x}^2 + \bar{y}^2} = \frac{r}{n}\sin\frac{n\alpha}{n-1}\operatorname{cosec}\frac{\alpha}{n-1}.$$

Writing this in the form

$$\frac{r}{a}\sin\left(1-\frac{1}{n-1}\right)a\,.\,\frac{\frac{a}{n-1}}{\sin\frac{a}{n-1}}\,.\,\frac{n-1}{a},$$

$$\text{or}\quad \frac{r}{a}\,.\,\sin\left(1-\frac{1}{n-1}\right)a\,.\,\frac{\frac{a}{n-1}}{\sin\frac{a}{n-1}}\,.\left(1-\frac{1}{n}\right),$$

we see that if the number of particles be indefinitely increased, their C. of G. will ultimately coincide with that of the circular arc. Now when $n$ is indefinitely great, $\frac{1}{n-1}$ and $\frac{1}{n}$ may be considered as indefinitely small compared with unity, and the value of $\frac{\frac{a}{n-1}}{\sin\frac{a}{n-1}}$ is ultimately equal to unity.

$\therefore$ if $G'$ be the C. of G. of the circular arc, $OG' = \frac{r}{a}\sin a$.

**7.** Let $O$ be the origin, $P$ the point $(h, k)$ $PA$, $PB$ perpendiculars on the axes of $x$ and $y$ respectively; $PN$, $PM$ are parallel to these axes, and $PQ$ is perpendicular to $AB$.

Then $OA = OM + MA = h + k\cos\omega$; $OB = ON + NB = k + h\cos\omega$.

The equation to $AB$ is $1 = \frac{x}{OA} + \frac{y}{OB} = \frac{1}{h + k\cos\omega} + \frac{1}{k + h\cos\omega}$,

or $x(h\cos\omega + k) + y(k\cos\omega + h) = (h + k\cos\omega)(k + h\cos\omega)$,

$$\therefore PQ^2 = \frac{\left\{(h\cos\omega+k)h+(k+h\cos\omega)k-(h+k\cos\omega)(k+h\cos\omega)\right\}^2}{(h\cos\omega+k)^2+(k+h\cos\omega)^2-2(h+k\cos\omega)(k+h\cos\omega)}\sin^2\omega$$

$$= \frac{h^2k^2(1-\cos^2\omega)^2\sin^2\omega}{(h^2+k^2+2hk\cos\omega)(1-\cos^2\omega)}.$$

$$\therefore PQ = \frac{hk\sin^2\omega}{\sqrt{h^2+k^2+2hk\cos\omega}}.$$

Suppose the equation to $PQ$ to be $Ax + By + C = 0$. (1).

Since this is perpendicular to $AB$, whose equation is

$$(h\cos\omega + k)x + (k\cos\omega + h)y - (h + k\cos\omega)(k\cos\omega + h) = 0,$$

$\therefore$ by *Salm. Con.* Art. 26, Cor. 2, we must have

$$A(h\cos\omega + k) + B(k\cos\omega + h) = \{A(h + k\cos\omega) + B(k\cos\omega + h)\}\cos\omega,$$

$$\therefore Ak = -Bh, \quad \therefore \frac{A}{h} = -\frac{B}{k}.$$

Since $PQ$ passes through $(h, k)$, (1) may be written

$$A(x - h) + B(y - k) = 0,$$

$$\text{or } h(x - h) - k(y - k) = 0, \quad \therefore hx - ky = h^2 - k^2.$$

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## Paper XLVIII.

1. The given expression

$$= (b-c)^2 \begin{vmatrix} (c-a)^2, & a^2 \\ a^2 & , (a-b)^2 \end{vmatrix} + c^2 \begin{vmatrix} a^2, (a-b)^2 \\ c^2, \quad b^2 \end{vmatrix} + b^2 \begin{vmatrix} c^2 & , b^2 \\ (c-a)^2, & a^2 \end{vmatrix}$$

$$= (b-c)^2\{(c-a)^2(a-b)^2 - a^4\} + c^2\{a^2b^2 - c^2(a-b)^2\} + b^2\{a^2c^2 - b^2(c-a)^2\}$$

$$= 2(u-bc)\{(b-c)^2(ac+ab-bc-2a^2) + c^2(ab+bc-ca) + b^2(ac+bc-ab)\}$$

$$= 4(u - bc)(b^2c^2 - a^2b^2 - a^2c^2 + 2a^2bc)$$

$$= 4(u - bc)\{b^2c^2 - a^2(b. - c)^2\}$$

$$= 4(u - bc)(bc + ab - ac)(bc + ca - ab)$$

$$= 16(u - bc)(u - ca)(u - ab).$$

2. Clear of fractions, and arrange in descending powers of $x$.

$$\therefore x^2\{a^2(b-c) + \beta^2(c-a) + \gamma^2(a-b)\} - x\{a^2(b^2-c^2) + \beta^2(c^2-a^2) + \gamma^2(a^2-b^2)\}$$
$$+ a^2bc(b - c) + \beta^2ca(c - a) + \gamma^2ab(a - b) = 0.$$

Since this equation has equal roots,

$$\therefore \{x^2(a^2-c^2) + \beta^2(c^2-a^2) + \gamma^2(a^2-b^2)\}^2$$
$$= 4\{a^2(b-c) + \beta^2(c-a) + \gamma^2(a-b)\}\{a^2bc(b-c) + \beta^2ca(c-a) + \gamma^2ab(a-b)\}$$

Arrange the result in terms of the powers and coefficients of $a$, $\beta$, $\gamma$.

$$\therefore a^4(b-c)^4 + \beta^4(c-a)^4 + \gamma^4(a-b)^4 - 2\beta^2\gamma^2(c-a)^2(a-b)^2 - 2\gamma^2a^2(a-b)^2(b-c)^2 - 2a^2\beta^2(b-c)^2(c-a)^2 = 0,$$

$$\therefore \pm a(b-c) \pm \beta(c-a) \pm \gamma(a-b) = 0.$$ See XIII. 1.

3. Let $S = a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + \ldots.$

$$C = a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \ldots.$$

$$\therefore C + Si = ae^{\theta i} + a^2e^{2\theta i} + \ldots.$$

$$= (1 - ae^{\theta i})^{-1} - 1,$$

$$\therefore (C + 1 + Si)(1 - ae^{\theta i}) = 1,$$

$$\therefore (C + 1 + Si)(1 - a\cos\theta - i\,.\,a\sin\theta) = 1.$$

Equating real and unreal parts

$$(C + 1)(1 - a\cos\theta) + S\,.\,a\sin\theta = 1,$$

$$(C + 1)\,.\,a\sin\theta - S(1 - a\cos\theta) = 0,$$

$$\therefore S(a^2\sin^2\theta + 1 - 2a\cos\theta + a^2\cos^2\theta) = a\sin\theta,$$

$$\therefore S = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}.$$

4. $$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\ldots.$$

$$\therefore e^{\theta i} - e^{-\theta i} = 2i\theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\ldots.$$

Let $\theta i = \pi$,

$$\therefore e^{\pi} - e^{-\pi} = 2\pi(1+1)\left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{3^2}\right)\ldots\ldots\ldots \quad (1)$$

Let $\theta i = \frac{\pi}{2}$,

$$\therefore e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}} = \pi\left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{4^2}\right)\ldots\ldots\ldots \quad (2)$$

$$= \pi\cdot\frac{5}{4}\cdot\frac{17}{16}\cdot\frac{37}{36}\cdot\ldots = y.$$

Divide (1) by (2).

$$\therefore e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}} = 2\left(1+1\right)\left(1+\frac{1}{3^2}\right)\ldots.$$

$$= 2 \cdot \frac{2}{1} \cdot \frac{10}{9} \cdot \frac{26}{25} \cdot \ldots = 2x,$$

$$\therefore 4x^2 - y^2 = (e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}})^2 - (e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}})^2 = 4.$$

**5.** Let $ABC$ be the right-angled triangle. On the hypotenuse $BC$ describe the square $BFGC$. Let $BG$, $CF$ intersect in $E$. Draw $ED$ and $EH$ perpendicular to $AC$, $AB$ respectively. Suppose $AC > AB$.

Since the angles at $A$, $H$, $D$ are right angles, $\therefore$ $HED$ is a right angle. The angle $BEC$ is a right angle, $\therefore$ $BEH = DEC$. $\therefore$ from the right-angled triangles $EBH$, $ECD$, since $EB = EC$, $\therefore$ $EH = ED$.

*** 6.** Let $OSO'S'$ be a square, and $OM$, $O'M'$, $SY$, $S'Y'$ perpendiculars from its angular points upon the variable straight line.

Then *Casey*, p. 24, $OM^2 + O'M'^2 - 2SY \cdot S'Y' =$ area of square. $\therefore$ if $OM^2 + O'M'^2$ constant $= k^2$, $SY \cdot S'Y'$ is also constant, and the variable straight line always touches an ellipse, foci $S$ and $S'$.

If $a$, $b$ be the semi-axes, $SY \cdot S'Y' = b^2 = a^2(1 - e^2)$, and area of square $= 2CS^2 = 2a^2e^2$,

$\therefore$ $k^2 - 2a^2(1-e^2) = 2a^2e^2$. $\therefore$ $k^2 = 2a^2$. $\therefore$ $(2a)^2 = 2k^2 = 2(OM^2 + O'M'^2)$.

Also, $O$ and $O'$ are on the line through the centre at right angles to the major axis, *i.e.* are on the conjugate axis, and $CO = CO' = CS$.

*** 7.** Let $C\alpha$, $C\beta$ be the equiconjugates, and let the tangent at $P$ meet $C\alpha$ in $Q$ and $C\beta$ in $R$. Draw $P\lambda$, $P\mu$ parallel to $C\alpha$ and $C\beta$. Bisect $RQ$ in $D$, and draw $DL$ parallel to $C\alpha$.

Then $$C\mu \cdot CQ = C\alpha^2 = C\beta^2 = C\lambda \cdot CR,$$

$$\therefore C\mu : C\lambda :: CR : CQ :: RL : LD :: CL : LD,$$

$$\therefore C\mu : P\mu :: CL : LD.$$ And the angle $P\mu C = CLD$.

$\therefore$ the triangles $PC\mu$, $DCL$ are similar, $\therefore$ the angle $PC\mu = DCL$.

*i.e.* $CP$ and $CD$ are equally inclined to the equiconjugates, and $\therefore$ also to the major axis. Now if we produce $QC$ to $Q'$ so that $CQ' = CQ$, $RQ'$ is a tangent to the ellipse, and is also parallel to $CD$. $\therefore$ $CP$ and $RQ'$ are equally inclined to the major axis, as are also $CP$ and $MN$. $\therefore$ $RQ'$ is parallel to $MN$. Similarly it may be shewn that the tangent from $Q$ is parallel to $MN$.

## Paper XLIX.

1. $(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \ldots.$

$$(1+x)^n = x^n + \ldots. + \frac{\lfloor n}{\lfloor n-r \rfloor r} x^{n-r} + \ldots.$$

$$\therefore f_1(r) = \text{coef. } x^{n-r} \text{ in } (1+x)^{2n} = \text{coef. } x^{n+r},$$

$$\therefore (1+x)^{2n} = 1 + \ldots. + f_1(0)x^n + f_1(1)x^{n+1} + \ldots.$$

$$(1+x)^n = x^n + \ldots. + \frac{\lfloor n}{\lfloor n-r \rfloor r} x^{n-r} + \ldots.$$

$$\therefore f_2(r) = \text{coef. } x^{2n-r} \text{ in } (1+x)^{3n} = \text{coef. } x^{n+r},$$

$$\therefore (1+x)^{3n} = 1 + \ldots. + f_2(0)x^n + f_2(1) \cdot x^{n+1} + \ldots.$$

$$(1+x)^n = x^n + \ldots. + \frac{\lfloor n}{\lfloor n-r \rfloor r} x^{n-r} + \ldots.$$

$$\therefore f_3(r) = \text{coef. } x^{2n-r} \text{ in } (1+x)^{4n} = \text{coef. } x^{2n+r}.$$

$$\text{So } f_4(r) = \text{coef. } x^{3n-r} \text{ in } (1+x)^{5n} = \text{coef. } x^{2n+r},$$

$$f_5(r) = \text{coef. } x^{3n-r} \text{ in } (1+x)^{6n} = \text{coef. } x^{3n+r}.$$

. . . . .

$\therefore$ if $m$ be even, and $= 2p$,

$$f_m(r) = \text{coef. } x^{\frac{2(p+1)n}{2}-r} \text{ in } (1+x)^{(2p+1)n} = \text{coef. } x^{pn+r}$$

$$\therefore f_m(n) = \text{coef. } x^{pn} \text{ or coef. } x^{(p+1)n} \text{ in } (1+x)^{(2p+1)n} = \frac{\lfloor (2p+1)n}{\lfloor (p+1)n \rfloor pn}$$

$$= \frac{\lfloor (m+1)n}{\lfloor \frac{1}{2}(m+2)n \rfloor \frac{1}{2}mn}.$$

If $m$ be odd, and $= 2p+1$,

$$f_m(r) = \text{coef. } x^{(p+1)n+r} \text{ in } (1+x)^{2(p+1)n},$$

$$\therefore f_m(n) = \text{coef. } x^{(p+2)n} \text{ in } (1+x)^{2(p+1)n} = \frac{\lfloor 2(p+1)n}{\lfloor (p+2)n \rfloor pn}$$

$$= \frac{\lfloor (m+1)n}{\lfloor \frac{1}{2}(m+3)n \rfloor \frac{1}{2}(m-1)n}.$$

2. Consider $a$ quantities each $= \frac{1}{a}$, $b$ quantities each $= \frac{1}{b}$, $c$ quantities each $= \frac{1}{c}$.

$$\text{Their A.M.} = \frac{a \cdot \frac{1}{a} + b \cdot \frac{1}{b} + c \cdot \frac{1}{c}}{a + b + c} = \frac{3}{a + b + c}.$$

$$\text{Their G.M.} = \left(\frac{1}{a^a} \cdot \frac{1}{b^b} \cdot \frac{1}{c^c}\right)^{\frac{1}{a+b+c}}.$$

Since the A.M. $>$ the G.M.

$$\therefore \frac{3}{a + b + c} > \left(\frac{1}{a^a \cdot b^b \cdot c^c}\right)^{\frac{1}{a+b+c}}.$$

$$\therefore a^{\frac{a}{a+b+c}} \cdot b^{\frac{b}{a+b+c}} \cdot c^{\frac{c}{a+b+c}} > \tfrac{1}{3}(a + b + c).$$

This may be shewn to hold in a similar manner for $n$ quantities.

3. Let the angles referred to their respective units be $2x$, $x$, $3x$.

Then $$2x + \frac{100}{60}x + 3 \cdot \frac{10000}{3600}x = 180,$$

$\therefore x = 15$, and the angles are 30°, 25°, and 125°.

4. Since the product of $\tan^2\theta$ and $\cot^2\theta$ is constant, and $= 1$, $\therefore$ the sum is a minimum when $\tan^2\theta = \cot^2\theta = \sqrt{(\text{product})} = 1$,

$$\therefore \tan\theta = \pm 1, \quad \therefore \theta = \left(n \pm \frac{1}{4}\right)\pi.$$

5. Let $BC$, $BD$, $BE$ be the three given straight lines, $BD$ being between $BC$ and $BE$. Let the given point $A$ be on the side of $BE$ remote from the other two lines. In $BC$ take any point $C$, and draw $CH$ perpendicular to $BD$, and produce it to $K$ so that $CH$ is to $HK$ in the given ratio. Through $K$ draw $KE$ parallel to $BD$, cutting $BE$ in $E$, and join $CE$, meeting $BH$ in $D$. Through $A$ draw $ANML$ parallel to $CDE$, meeting $BE$, $BD$, $BC$ in $N$, $M$, $L$ respectively. Then $AL$ is the line required.

For $LM : MN :: CD : DE :: CH : HK$,

$\therefore LM : MN$ in the given ratio.

*6. Let $H$, $K$ be corresponding points in $AB$, $AD$, so that

$$AH : HB :: AK : KD$$

or
$$HB : KD :: AB : AD.$$

Let $EH$, $CK$ meet in $O$, and draw $OM$ parallel to $AB$ to meet $BC$ in $M$, and let $E$ lie between $B$ and $C$.

Then
$$OM : CM :: CD : KD,$$

$$OM : EM :: HB : BE,$$

$\therefore$ $OM^2 : CM.EM :: CD.HB : BE.KD :: CD.AB : BE.AD :: AB^2 : BE.BC.$

$\therefore$ the locus of $O$ is a hyperbola having $CE$ for diameter, and line bisecting $CE$ and parallel to $AB$ for the conjugate diameter. If $E$ lies on $CB$ produced, $M$ lies between $C$ and $E$, and the locus is an ellipse.

7. When two conics have double contact, their equations can be put in the form $S = 0$, and $S - Ka^2 = 0$, and their difference is evidently a perfect square. $\therefore$ in the question we must have

$x^2(a - \alpha) + 2xy(c - \gamma) + y^2(b - \beta) + 2x(a' - \alpha')$ a perfect square.

$\therefore$ all the terms involving $x$ and $y$ must be of the 2nd degree.

This gives us the two conditions

$$a' - \alpha' = 0; \quad (a - \alpha)(b - \beta) = (c - \gamma)^2.$$

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## Paper L.

*1. $\xi = lx + my + nz$; (1) $\quad x = l\xi + m\eta + n\zeta$; (4)

$\eta = nx + ly + mz$; (2) $\quad y = n\xi + l\eta + m\zeta$; (5)

$\zeta = mx + ny + lz$; (3) $\quad z = m\xi + n\eta + l\zeta$; (6)

From (1), (2), (3), (4) we have

$$x = l(lx + my + nz) + m(nx + ly + mz) + n(mx + ny + lz)$$
$$= (l^2 + 2mn)x + (n^2 + 2lm)y + (m^2 + 2ln)z.$$

Since this is true for all values of $x$, $y$, $z$,

$\therefore$ $l^2 + 2mn = 1$; $(A)$ $\quad m^2 + 2nl = 0$; $(B)$ $\quad n^2 + 2lm = 0$; $(C)$

and the same three conditions are obtained by expressing $y$ or $z$ in terms of $x$, $y$, $z$.

From $(B)$ and $(C)$ $\quad m^2 - n^2 - 2l(m - n) = 0,$

$$\therefore (m - n)(m + n - 2l) = 0.$$

From $(B)$ and $(C)$ we deduce that when $m$ and $n$ are real, they both have their sign opposite to that of $l$.

$$\therefore\ m + n - 2l \neq 0. \quad \therefore\ m - n = 0.$$

$\therefore$ excluding zero values, we have $2l + m = 0$, and $l^2 + 2m^2 = 1$.

$$\therefore\ l = \pm \tfrac{1}{3}. \quad \therefore\ m = n = -2l = \pm \tfrac{2}{3}.$$ *See* Errata.

2. Assume that

$$\frac{x^2 + px + q}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c},$$

where $A$, $B$, and $C$ are constants.

Then

$$x^2 + px + q \equiv A(x-b)(x-c) + B(x-c)(x-a) + C(x-a)(x-b).$$

Since this is always true whatever be the value of $x$, we may put $x = a$.

$$\therefore\ a^2 + pa + q = A(a-b)(a-c).$$

Similarly by putting $x = b$, and $x = c$, we get

$$b^2 + pb + q = B(b-c)(b-a); \quad c^2 + pc + q = C(c-a)(c-b).$$

Thus $A$, $B$, and $C$ are determined.

Now
$$\frac{A}{x-a} = -\frac{A}{a}\left(1 - \frac{x}{a}\right)^{-1}.$$

$\therefore$ the term involving $x^n$ in this expression is $-\dfrac{A}{a^{n+1}}$.

$\therefore$ the general term required is

$$\frac{1}{a^{n+1}} \cdot \frac{a^2 + pa + q}{(a-b)(c-a)} + \frac{1}{b^{n+1}} \cdot \frac{b^2 + bp + q}{(a-b)(b-c)} + \frac{1}{c^{n+1}} \cdot \frac{c^2 + pc + q}{(b-c)(c-a)}.$$

3. The given expression

$$= \frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B}$$

$$= \frac{\sin A \cos A + \sin B \cos B + \sin C \cos C}{\sin A \sin B \sin C}$$

$$= \frac{\sin 2A + \sin 2B + \sin 2C}{2 \sin A \sin B \sin C}, \text{ and } A + B + C = 180°,$$

$$= \frac{4 \sin A \sin B \sin C}{2 \sin A \sin B \sin C} = 2.$$

4. Let $E$, $F$ be the points of contact on $CB$, $CA$ of the escribed circle opposite $C$; $E'$, $F'$ corresponding points on $BC$, $BA$ for the escribed circle opposite $B$. From $A$ draw $AH$ perpendicular to $BC$, meeting $EF$ produced in $G$, and $E'F'$ produced in $G'$. We will first prove that

$$(s - b\cos C)\cot\frac{C}{2} = (s - c\cos B)\cot\frac{B}{2},$$

$$(s - b)\cot\frac{C}{2} = \frac{S}{s-a} = (s - c)\cot\frac{B}{2} \quad . \quad . \quad . \quad . \quad (1)$$

$$b\sin C = c\sin B\,; \quad \therefore 2b\,.\,\sin^2\frac{C}{2}\cot\frac{C}{2} = 2c\,.\,\sin^2\frac{B}{2}\cot\frac{B}{2};$$

$$\therefore b(1 - \cos C)\cot\frac{C}{2} = c(1 - \cos B)\cot\frac{B}{2} \quad . \quad . \quad . \quad (2)$$

$\therefore$ adding (1) and (2) we have

$$(s - b\cos C)\cot\frac{C}{2} = (s - c\cos B)\cot\frac{B}{2}.$$

Now $GH = HE\tan HEG = (CE - CH)\cot DEF$

$= (s - b\cos C)\cot\frac{C}{2}$, since $DECF$ can be inscribed in a circle,

$= (s - c\cos B)\cot\frac{B}{2}$

$= G'H$. $\therefore$ $G'$ coincides with $G$.

$\therefore$ $EF$ and $E'F'$ intersect $AH$ in the same point $G$.

Again $AG = GH - AH = (s - c\cos B)\cot\frac{B}{2} - c\sin B$

$= (s - c\cos B - c\,.\,2\sin^2\frac{B}{2})\cot\frac{B}{2}$

$= (s - c)\cot\frac{B}{2} = r_a$.

5. Join $EB$. Then the angle $EAB = CAD$, and $AEB = ACD$, $\therefore$ the triangles $ABE$, $ACD$ are similar,

$$\therefore AB : AE :: AD : AC, \quad \therefore AB\,.\,AC = AE\,.\,AD,$$

$$\therefore AB\,.\,AC + AD^2 = AD(AE + AD) = AD\,.\,ED = DC\,.\,DB$$

6. Let $CP$, $CD$ be the equiconjugate diameters, and $QCQ'$ the diameter perpendicular to $CP$. Let $QVq$ be the chord through $Q$ parallel to $CD$. Then $Qq$ is the double ordinate of $CP$. Let the tangent at $q$ meet $CP$ produced in $T$.

Then $CV \cdot CT = CP^2$, $\therefore\ CV \cdot VT = CP^2 - CV^2 = QV^2 = QV \cdot Vq$.

$\therefore$ a circle will go round $QCqT$, and the angle $QqT = QCT =$ a right angle. $\therefore$ $Qq$ is a normal at $q$. Similarly it may be shewn that the chord $Q'q'$ is normal at $q'$.

7. Let $\alpha, \beta, \gamma$ be the eccentric angles of $L$, $M$, $N$, $\delta$ that of $P$. Then by *Salm. Con.* Art. 244, Ex. 5, $\alpha = 240° - \frac{1}{3}\delta$, $\beta = 120° - \frac{1}{3}\delta$, $\gamma = -\frac{1}{3}\delta$. And by Art. 231, Ex. 5, the area of the triangle formed by $\alpha, \beta, \gamma$

$$= 2ab \sin \tfrac{1}{2}(\alpha - \beta) \sin \tfrac{1}{2}(\beta - \gamma) \sin \tfrac{1}{2}(\alpha - \gamma)$$

$$= 2ab \cdot \sin 120° \sin 120° \sin 240° = \frac{3\sqrt{3}}{4} ab = \text{const.}$$

And XXXVII. 6, $\frac{3\sqrt{3}}{4} ab =$ the area of the maximum triangle which can be inscribed in the ellipse.

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## Paper LI.

1. (1) From (1) and (2),

$$\frac{x}{(c+a)(a-b) - (c-a)(a+b)} = \frac{y}{(a+b)(b-c) - (a-b)(b+c)}$$

$$= \frac{z}{(b+c)(c-a) - (b-c)(c+a)},$$

$$\therefore \frac{x}{bc - a^2} = \frac{y}{ca - b^2} = \frac{z}{ab - c^2} = R \text{ suppose.}$$

From (3) by simplifying we get

$$\frac{abx}{bc - a^2} + \frac{bcy}{ca - b^2} + \frac{caz}{ab - c^2} = n(bc + ca + ab),$$

$$\therefore R(bc + ca + ab) = n(bc + ca + ab), \quad \therefore R = n,$$

$$\therefore x = n(bc - a^2)\,;\ y = n(ca - b^2)\,;\ z = n(ab - c^2).$$

(2) Put $y = b - a + x$. Then the equation becomes

$$\sqrt{(y+b)(y-a)} + \sqrt{(y-b)(y+a)} = \sqrt{(y+a)(y+b) - 4ab}, \quad (1)$$

and $(y+b)(y-a) - (y-b)(y+a) = 2(b-a)y,$

$$\therefore \sqrt{(y+b)(y-a)} - \sqrt{(y-b)(y+a)} = \frac{2(b-a)y}{\sqrt{(y+a)(y+b) - 4ab}}. \quad (2)$$

Adding (1) and (2)

$$2\sqrt{(y+b)(y-a)} = \sqrt{(y+a)(y+b) - 4ab} + \frac{2(b-a)y}{\sqrt{(y+a)(y+b) - 4ab}}$$

$$= \frac{(y-a)(y+3b)}{\sqrt{(y+a)(y+b) - 4ab}},$$

$$\therefore \text{ either } y = a, \text{ and } x = 2a - b,$$

$$\text{or } 4(y+b)\{y^2 + (a+b)y - 3ab\} = (y-a)(y+3b)^2,$$

$$\therefore 3y^3 + (5a + 2b)y^2 - (2ab + 5b^2)y - 3ab^2 = 0,$$

$$\therefore (y-b)\{3y^2 + 5(a+b)y + 3ab\} = 0.$$

This gives us the 3 other values of $y$, and solving, we get finally

$$x = 2a - b,\ a,\ \tfrac{1}{6}(a - 11b) \pm \sqrt{25(a^2 + b^2) + 14ab}.$$

(3) $x^2y^2(x^4 - y^4) = a^2 = x^2y^2(x^2 + y^2)^2(xy^3 - 1)^2,$

$\therefore x^2 - y^2 = (x^2 + y^2)(x^2y^6 - 2xy^3 + 1)$, neglecting the factors $x^2y^2(x^2 + y^2)$, which do not give admissible values of $x$ and $y$ when equated to zero.

$$\therefore x^2 - y^2 = x^4y^6 + x^2y^8 - 2x^3y^3 - 2xy^5 + x^2 + y^2,$$

$$\therefore x^4y^4 + x^2y^6 - 2x^3y - 2xy^3 = -2,$$

From (1)
$$\left.\begin{aligned} \therefore xy(xy^3 - 2)(x^2 + y^2) &= -2 \\ xy(xy^3 - 1)(x^2 + y^2) &= a \end{aligned}\right\} (A)$$

$$\therefore a(xy^3 - 2) = -2(xy^3 - 1),$$

$$\therefore xy^3 = \frac{2(a+1)}{a+2}, \quad \therefore xy^3 - 1 = \frac{a}{a+2},$$

$\therefore$ from $(A)$ by substituting the value of $xy^3 - 1$

$$xy(x^2 + y^2) = a + 2,$$

and
$$x^2y^2(x^4 - y^4) = a^2,$$

$$\therefore xy(x^2 - y^2) = \frac{a^2}{a+2},$$

$$\therefore (x^2 - y^2)(a + 2)^2 = (x^2 + y^2)a^2,$$

$$\therefore 2x^2(a + 1) = y^2(a^2 + 2a + 2).$$

Now $$\frac{4(a + 1)^2}{(a + 2)^2} = x^2y^6 = y^8 \cdot \frac{a^2 + 2a + 2}{2(a + 1)},$$

$$\therefore y^8 = \frac{8(a + 1)^3}{(a + 2)^2(a^2 + 2a + 2)};$$

$$x^8 = y^8 \cdot \frac{(a^2 + 2a + 2)^4}{16(a + 1)^4} = \frac{(a^2 + 2a + 2)^3}{2(a + 1)(a + 2)^2}.$$

2. Let $4l + a$, $4m + b$, $4n + c$ be 3 numbers, in which $a$, $b$, $c$ are all numbers $< 4$. The sum of their squares is

$$16(l^2 + m^2 + n^2) + 8(al + bm + cn) + a^2 + b^2 + c^2 = 8p + a^2 + b^2 + c^2.$$

Now if this is of the form $8x + 7$, we must have

$$a^2 + b^2 + c^2 = 8y + 7.$$

For this to be possible, we must have either one or three of the numbers $a$, $b$, $c$, odd. Now all the sets of 1 odd and 2 even, or 3 odd numbers less than 4 are

1, 1, 1; 3, 3, 3; 1, 1, 3; 1, 2, 2; 1, 2, 0; 1, 0, 0; 3, 2, 2; 3, 2, 0; 3, 0, 0; 1, 3, 3.

And we see that in none of these cases is $a^2 + b^2 + c^2$ of the form $8y + 7$.

3. Tripos 1875. 2nd Tuesday morning. No. 2.

4. Take the common centre as origin, and a line through it parallel to $ABC$ as the axis of $x$. Let the coordinates of $A$, $B$, $C$ be $(\alpha, p)$, $(\beta, p)$, $(\gamma, p)$, and let $a$, $b$, $c$ be the radii of the circles.

Then $a^2 = \alpha^2 + p^2$; $b^2 = \beta^2 + p^2$; $c^2 = \gamma^2 + p^2$.

The equations to the tangents at $A$, $B$, $C$ are

$$\alpha x + py = a^2; \quad \beta x + py = b^2; \quad \gamma x + py = c^2.$$

$\therefore$ by *Salm. Con.* Art. 39, the area contained by these lines is

$$\frac{\{a(-c^2p + b^2p) + \beta(-a^2p + c^2p) + \gamma(-b^2p + a^2p)\}^2}{2(\alpha p - \beta p)(\beta p - \gamma p)(\gamma p - \alpha p)}$$

$$= \frac{\{a(\beta^2 - \gamma^2) + \beta(\gamma^2 - \alpha^2) + \gamma(\alpha^2 - \beta^2)\}^2}{2p(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)}$$

$$= \frac{(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)}{2p} = \frac{AB \cdot BC \cdot CA}{2p}.$$

5. Tripos 1875. Monday morning. No. 9.

6. Let the triangle $DEF$ be inscribed in $ABC$ so that $D$ is opposite $A$, &c. Then the triangle $FDE = FAE$, each being one fourth of $ABC$. Draw $AG$, $DH$ perpendicular to $EF$, and let $EF$ and $AD$ intersect in $K$. Then since $AEF = DEF$, and they are on equal bases, $\therefore$ their altitudes are equal. $\therefore AG = DF$. $\therefore$ from the right-angled triangles $AGK$, $DHK$ it follows that $AK = KD$.

7. Since the line joining two of the points mentioned is at right angles to the line joining the other two

$$\therefore \frac{y_2 - y_1}{x_2 - x_1} \cdot \frac{y_4 - y_3}{x_4 - x_3} + 1 = 0,$$

$$\therefore (y_2 - y_1)(y_4 - y_3) + (x_2 - x_1)(x_4 - x_3) = 0.$$

But $$y_1 = \frac{c^2}{x_1} \text{ \&c.}$$

$$\therefore \frac{c^2}{x_1 x_2}(x_1 - x_2) \cdot \frac{c^2}{x_3 x_4}(x_3 - x_4) + (x_2 - x_1)(x_4 - x_3) = 0,$$

$$\therefore x_1 x_2 x_3 x_4 + c^4 = 0.$$

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## Paper LII.

*1. The given expression

$$= 3^{2n} \cdot 5^{2n} + 5^{4n} - 35 \cdot 5^{2n} - 3^{2n} + 34$$
$$= 3^{2n}(5^{2n} - 1) + 5^{4n} - 1 - 35(5^{2n} - 1)$$
$$= (5^{2n} - 1)(3^{2n} + 5^{2n} - 34).$$

Now $5^{2n} - 1 = 25^n - 1 = (1 + 24)^n - 1 = M(24) = M(2^3 \times 3)$.

Also $3^{2n} + 5^{2n} - 34 = 3^{2n} + M(24) + 1 - 34$
$= 3^{2n} + M(24) - 33 = M(3)$.

And $3^{2n} + 5^{2n} - 34 = (4 - 1)^{2n} + (4 + 1)^{2n} - 34$
$= 2\left\{1 + \frac{2n(2n-1)}{1 \cdot 2} \cdot 16 + M(256)\right\} - 34$
$= 32n(2n - 1) + M(256) - 32$
$= M(32) = M(2^5)$.

$\therefore 3^{2n} + 5^{2n} - 34 = M(3 \times 2^5)$.

$\therefore$ the given expression $= M(2^8 \times 3^2) = M(2304)$. *See* Errata.

2. $ADE = \frac{\pi}{2} - A$ ; $AE = c\cos A$ ; $p_4 = -r' = -2R\cos A\cos B\cos C$.
*Todh. Trig.* cap. xvi. Ex. 27; $\Delta = 2R^2 \sin A \sin B \sin C$

$$p_1 = AE\sin B = c\cos A\sin B = 2R\sin B\sin C\cos A,$$

$$\therefore\ p_1+p_2+p_3 = 2R(\sin B\sin C\cos A + \sin C\sin A\cos B + \sin A\sin B\cos C)$$

$$= 2R\left\{\sin C\sin(A+B) + \tfrac{1}{2}(\cos\overline{A-B} - \cos\overline{A+B})\cos C\right\}$$

$$= 2R\left\{1 - \cos^2 C + \tfrac{1}{2}(\cos\overline{A-B} - \cos\overline{A+B})\cos C\right\}$$

$$= 2R\left\{1 + \tfrac{1}{2}\cos C(\cos\overline{A-B} + \cos\overline{A+B})\right\}$$

$$= 2R(1 + \cos A\cos B\cos C);$$

$$\therefore\ p_1+p_2+p_3+p_4 = 2R.$$

$$p_1p_2 + p_2p_3 + p_3p_1 = 4R^2\sin A\sin B\sin C(\cos A\cos B\sin C + \ldots)$$

$$= 4R^2\sin A\sin B\sin C\left\{\cos A\sin\overline{B+C} + \sin A\cos B\cos C\right\}$$

$$= 4R^2\sin^2 A\sin B\sin C\left\{\cos B\cos C - \cos\overline{B+C}\right\}$$

$$= 4R^2\sin^2 A\sin^2 B\sin^2 C = \frac{\Delta^2}{R^2};$$

and $$p_4(p_1+p_2+p_3) = -r'(2R+r') = -2Rr' - r'^2,$$

$$\therefore \text{ sum of products 2 together} = \frac{\Delta^2}{R^2} - 2Rr' - r'^2,$$

$$p_1p_2p_3 = 4R^2\sin^2 A\sin^2 B\sin^2 C\,.\,2R\cos A\cos B\cos C$$

$$= \frac{\Delta^2}{R^2}\,.\,r'.$$

$$p_4(p_1p_2 + p_2p_3 + p_3p_1) = -r'\,.\,\frac{\Delta^2}{R^2},$$

$$\therefore \text{ sum of products 3 together} = 0.$$

$$p_1p_2p_3p_4 = -r'^2\,.\,\frac{\Delta^2}{R^2}$$

$\therefore\ p_1p_2p_3p_4$ are the roots of the equation

$$x^4 - 2Rx^3 + \left(\frac{\Delta^2}{R^2} - 2Rr' - r'^2\right)x^2 - \frac{\Delta^2}{R^2}\,.\,r'^2 = 0.$$

3. $\lfloor n+1+2$ is divisible by 2; $\lfloor n+1+3$ is divisible by 3; $\lfloor n+1+4$ is divisible by 4 . . . $\lfloor n+1+n+1$ is divisible by $n+1$.

Now the numbers on the left are consecutive numbers, each of which has a factor, and $\therefore$ is not a prime.

4. Let $PT$ be the tangent to the other circle. Then $PQ \cdot Pq = PT^2$ Now the chord of intersection is the radical axis. $\therefore$ the result follows by Casey, p. 113, Cor. 1.

5. Let $A$, $B$ be the centres of the circles, $C$ and $F$ their points of intersection. Let $ECD$ be the position of the chord required such that rect. $EC \cdot CD$ is a max., and let $E'CD'$ be a consecutive chord through $C$. Then since the rate of change of the rectangle is zero

$$\therefore \text{ rect. } EC \cdot CD = \text{ rect. } E'C \cdot CD'.$$

$\therefore$ a circle will go through $DD'EE'$. And since $D$, $D'$ are consecutive points on the one circle, as are also $E$, $E'$ on the other, we see that the line required is the chord of contact of the circle which touches the two given circles, the circle being such that its chord of contact passes through $C$.

To draw this chord, we have only to notice that it makes equal angles with the tangents at $E$ and $D$, and $\therefore$ also with the tangents at $C$. $\therefore$ the required chord bisects the angle between the tangents at the point of intersection.

*6. Let the diameters through $Q$, $R$ meet $PS$ in $M$ and $N$ respectively. Produce $PR$ to meet $SK$ in $T$, and $NR$ to meet $PK$ in $V$.

Then $$SN \cdot NP : SM \cdot MP :: RN : QM,$$

and $$MP : NP :: QM : VN,$$

$$\therefore SN : SM :: RN : VN,$$

$$\therefore SR : SL :: ST : SK.$$

$\therefore$ $PR$ is parallel to $KL$.

This problem can also be solved by Pascal's hexagon. Thus, if $\omega$ be the point at infinity in which the axis again meets the curve, and $\omega\omega$ denote the tangent at $\infty$, *i.e.* the line at $\infty$, the theorem tells us that the points of intersection of the following pairs of lines are collinear, viz. $(PQ, S\omega)$ or $K$; $(Q\omega, SR)$ or $L$, and $(\omega\omega, PR)$ or the point at infinity on $PR$. $\therefore$ $KL$ is parallel to $PR$.

7. Let the tangent at $P$ meet the tangents at $A$ and $A'$ in $T$ and $T'$. Let $ST'$, $S'T$ intersect in $Q$. Join $PQ$.

Since $TS$ bisects $PSA$, and $T'S$ bisects $PSA'$, $\therefore TST'$ is a right angle.

Similarly $TS'T'$ is a right angle. $\therefore$ a circle will go round $TSS'T'$.

Again $SQT = ST'T + S'TT' = ST'T + S'ST' = ST'T + TSP = SPT$.

$\therefore$ a circle will go round $SQPT$. And since $TSQ$ is a right angle, $\therefore TPQ$ is a right angle. $\therefore PQ$ is normal at $P$.

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## PAPER LIII.

1. (1) Employing the method of p. 19, No. II., we have

$$S = 1 - 2 - 3 - 2 + 1 + 6 + \dots$$
$$S_1 = \quad - 3 - 1 + 1 + 3 + 5 + \dots$$
$$S_2 = \quad\quad 2 + 2 + 2 + 2 + \dots$$

$\therefore$ the $n^{\text{th}}$ term is of the 2nd degree in $n$, and may be written in either of the forms

$$An^2 + Bn + C \dots (1) \quad \text{or} \quad A'n(n+1) + B'n + C' \dots (2)$$

If we take (1), and put $n = 1, 2, 3$ in succession, we have

$$A + B + C = 1; \quad 4A + 2B + C = -2; \quad 9A + 3B + C = -3.$$

From these equations we get $A = 1$, $B = -6$, $C = 6$,

$\therefore$ the $n^{\text{th}}$ term is $n^2 - 6n + 6$.

$\therefore$ by p. 18, Art. 26, the sum of $n$ terms

$$= \frac{n(n+1)(2n+1)}{6} - 6 \cdot \frac{n(n+1)}{2} + 6n.$$

If we take (2), we have

$$2A' + B' + C' = 1; \quad 6A' + 2B' + C' = -2; \quad 12A' + 3B' + C' = -3.$$

From these we find $A' = 1$, $B' = -7$, $C' = 6$,

$\therefore$ the $n^{\text{th}}$ term is $n(n+1) - 7n + 6$,

$\therefore$ sum of $n$ terms $= \frac{n(n+1)(n+2)}{3} - 7 \cdot \frac{n(n+1)}{2} + 6n.$

Of course the two results thus formed are the same, and merely differ in form.

(2) $S = 2 + 5 + 12 + 31 + 86 + 249 + \ldots$

$S_1 = \quad 3 + 7 + 19 + 55 + 163 + \ldots$

$S_2 = \quad\quad 4 + 12 + 36 + 108 + \ldots$

Here the second difference series gives us a geometrical progression, common ratio 3.

$\therefore$ p. 20, No. III., the $n^{\text{th}}$ term is of the form $A + Bn + C.3^{n-1}$.

Putting $n = 1, 2, 3$ in succession

$$A + B + C = 2; \quad A + 2B + 3C = 5; \quad A + 3B + 9C = 12.$$

These equations give $A = 0$, $B = 1$, $C = 1$.

$\therefore$ the $n^{\text{th}}$ term is $n + 3^{n-1}$.

$$\therefore \text{ sum of } n \text{ terms} = \frac{n(n+1)}{2} + \frac{3^n - 1}{3 - 1}.$$

$$= \tfrac{1}{2}\left\{n(n+1) + 3^n - 1\right\}.$$

(3) $$\frac{2}{1.4.7} + \frac{5}{4.7.10} + \frac{8}{7.10.13} + \ldots$$

$$\text{The } n^{\text{th}} \text{ term} = \frac{3n-1}{(3n-2)(3n+1)(3n+4)} = \frac{3n-2+1}{(3n-2)(3n+1)(3n+4)}$$

$$= \frac{1}{(3n+1)(3n+4)} + \frac{1}{(3n-2)(3n+1)(3n+4)}.$$

To write down the sum of $n$ terms of this series, we use the following rule.

Strike out the first factor of the $n^{\text{th}}$ term. Write the result negatively and divide by the product of the coef. of $n$ and the number of factors which remain. Then add a constant which can be determined by giving a suitable value to $n$.

$$\therefore \text{ sum of } n \text{ terms} = -\frac{1}{3}\cdot\frac{1}{3n+4} - \frac{1}{2.3}\frac{1}{(3n+1)(3n+4)} + C.$$

To determine $C$, we observe that when $n = 0$ the sum $= 0$,

$$\therefore C - \frac{1}{12} - \frac{1}{24} = 0, \quad \therefore C = \frac{1}{8}.$$

2. $(2n)^2 = 4n^2 = 4\left\{1 + 3 + 5 + \ldots + (2n-1)\right\}$

$= 4 + 12 + 20 + \ldots + 8n - 4$

$=$ an A.P. with common difference 8.

$$(2n+1)^2 = 4n^2 + 4n + 1$$
$$= 1 + 4 + 12 + 20 + \ldots + 8n - 4$$
$$+ 4 + 4 + 4 + \ldots + 4$$
$$= 1 + 8 + 16 + 24 + \ldots + 8n$$
$$= 1 + \text{an A.P. with common difference 8.}$$

3. Let $A$ be the summit, $B$ the foot of the hill, $D$ the point in $AB$ such that $AD = 3.DB$, $C$ the point in the plane. Then by the question, $CD$ bisects the angle $ACB$.

$$\therefore \frac{1}{3} = \frac{BD}{DA} = \frac{BC}{CA} = \frac{\sin(\theta - 30°)}{\sin\theta} = \frac{\sqrt{3}}{2} - \frac{1}{2}\cot\theta. \quad \therefore \tan\theta = \frac{3}{3\sqrt{3}-2}.$$

4. Let $ABC$ be a triangle. Produce $BA$ to $E$, and bisect the angle $CAE$ by $AD$, meeting the circum-circle in $D$. Join $DB$, $DC$.

Then the angle $DCB = EAD = DAC = DBC$. $\therefore DB = DC$.

Again, let $O_2$, $O_3$ be the centres of the escribed circles opposite $B$ and $C$.

Then $O_2BO_3$, $O_2CO_3$ are right angles, and $DB = DC$,

$\therefore$ the circle on $O_2O_3$ as diameter passes through $B$ and $C$, and $D$ is the centre. $\therefore$ $D$ is equidistant from $B$, $C$, $O_2$, $O_3$.

*5. Let $O_1$, $O_2$ be the middle points of $AC$, $BC$. Join $EO_1$, $ED$, $FO_2$, $FD$.

Since $AD = DB = O_1A + O_2B$, $\therefore O_1D = O_2C = O_2F$; and $O_2D = O_1C = O_1E$, and $DE = DF$, $\therefore$ the triangles $O_1DE$, $O_2DF$ have corresponding sides equal; $\therefore$ the angle $DO_1E = DO_2F$. $\therefore O_1E$, $O_2F$ are parallel. $\therefore EF$ passes through $C$, the centre of similitude of the two circles.

6. Let $O$ be the circum-centre, $P$ the orthocentre of $ABC$. Draw $PY$, $OY'$ perpendicular to $BC$. Then since $OAB = PAC$, &c. $\therefore$ $O$ and $P$ are the foci of the inscribed ellipse.

$$\therefore PY \cdot OY' = (\tfrac{1}{2}\text{ minor axis})^2.$$

$\therefore$ the product is the same for all the sides.

This can also be proved as follows:

$$PY = YC\tan PCY = AC\cos C\cot B = \frac{b}{\sin B}\cdot\cos B\cos C$$
$$= 2R\cos B\cos C,$$
$$OY' = R\cos A.$$

$\therefore PY \cdot OY' = 2R^2\cos A\cos B\cos C$, a symmetrical expression.

7. Let $ABC$, $PQR$ be the circum- and inscribed triangles. Let $p$, $q$, $r$ be the points of contact of $BC$, $CA$, $AB$. Through $P$, $Q$, $R$, $q$, $r$ draw diameters meeting $BC$ in $L$, $M$, $N$, $m$, $n$, respectively.

The diameters at $p$, $q$, $r$ will bisect the chords $QR$, $RP$, $PQ$.

Then $QR = MN = ML + LN = 2Ln + 2Lm = 2mn = 2rq = 4BC$.

Similarly it may be shewn that $RP = 4CA$, $PQ = 4AB$.

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## Paper LIV.

1. (1)
$$\begin{aligned} S &= 4 + 18 + 48 + 100 + 180 + 294 + \ldots \\ S_1 &= \quad 14 + 30 + 52 + 80 + 114 + \ldots \\ S_2 &= \quad\quad 16 + 22 + 28 + 34 + \ldots \\ S_3 &= \quad\quad\quad 6 + 6 + 6 + \ldots \end{aligned}$$

Here the 3rd difference series gives us a series of equal terms. $\therefore$ the $n^{\text{th}}$ term is of the 3rd degree in $n$, and is of the form

$$A + Bn + Cn(n+1) + Dn(n+1)(n+2).$$

Writing $n = 1, 2, 3, 4$ in succession, we have

$$\left.\begin{aligned} A + B + 2C + 6D &= 4 \\ A + 2B + 6C + 24D &= 18 \\ A + 3B + 12C + 60D &= 48 \\ A + 4B + 20C + 120D &= 100 \end{aligned}\right\}$$

Attending to the remark given in No. II. on p. 20, we find $A = 0$, $B = 0$, $C = -1$, $D = 1$,

$\therefore$ the $n^{\text{th}}$ term is $n(n+1)(n+2) - n(n+1)$.

When the $n^{\text{th}}$ term is of the above form, to obtain the sum of $n$ terms we have only to introduce an additional factor at the end of the $n^{\text{th}}$ term and divide by the number of factors thus obtained.

$$\therefore \text{ sum of } n \text{ terms} = \frac{n(n+1)(n+2)(n+3)}{4} - \frac{n(n+1)(n+2)}{3}.$$

$$= \frac{1}{12} n(n+1)(n+2)(3n+5).$$

(2) $$\begin{aligned} S &= 5 + 11 + 22 + 41 + 74 + 133 + \ldots \\ S_1 &= \quad 6 + 11 + 19 + 33 + 59 + \ldots \\ S_2 &= \quad\quad 5 + 8 + 14 + 26 + \ldots \\ S_3 &= \quad\quad\quad 3 + 6 + 12 + \ldots \end{aligned}$$

Here the 3rd difference series gives us a geometrical progression, common ratio 2.

$\therefore$ p. 20, No. III., the $n^{\text{th}}$ term is of the form

$$A + Bn + Cn\,(n+1) + D \cdot 2^{n-1}.$$

Putting $n$ in succession $= 1, 2, 3, 4$ we have

$$\left.\begin{aligned} A + B + 2C + D &= 5 \\ A + 2B + 6C + 2D &= 11 \\ A + 3B + 12C + 4D &= 22 \\ A + 4B + 20C + 8D &= 41 \end{aligned}\right\}$$

From these we find $A = 1,\ B = -1,\ C = 1,\ D = 3.$

$\therefore$ the $n^{\text{th}}$ term is $n\,(n+1) - n + 1 + 3 \cdot 2^{n-1}$

$\therefore$ the sum of $n$ terms $= \dfrac{n\,(n+1)\,(n+2)}{3} - \dfrac{n\,(n+1)}{2} + n + 3\,(2^n - 1).$

(3) Consider the series formed by the numerators

$$\begin{aligned} S &= 5 + 8 + 15 + 34 + 89 + 252 + \ldots \\ S_1 &= \quad 3 + 7 + 19 + 55 + 163 + \ldots \\ S_2 &= \quad\quad 4 + 12 + 36 + 108 + \ldots \end{aligned}$$

$\therefore$ the $n^{\text{th}}$ term is of the form $A + Bn + C \cdot 3^{n-1}.$

Putting $n = 1, 2, 3$ in succession we have

$$A + B + C = 5;\quad A + 2B + 3C = 8;\quad A + 3B + 9C = 15.$$

From these equations we find $A = 3,\ B = 1,\ C = 1,$

$\therefore$ the $n^{\text{th}}$ term of the original series is

$$\frac{3^{n-1} + n + 3}{n(n+1)\,(n+2)} \cdot \frac{1}{3^{n-1}} = \frac{1}{n(n+1)\,(n+2)} + \frac{n+3}{n(n+1)\,(n+2)} \cdot \frac{1}{3^{n-1}}.$$

As on p. 23, No. 5, assume

$$\frac{n+3}{n(n+1)(n+2)}\cdot\frac{1}{3^{n-1}} \equiv \frac{An+B}{(n+1)(n+2)}\cdot\frac{1}{3^{n-1}} - \frac{A(n-1)+B}{n(n+1)}\cdot\frac{1}{3^{n-2}},$$

$$\therefore n+3 \equiv -2An^2 - (3A+2B)n + 6(A-B).$$

Equating coefs. of like powers of $n$ on both sides, we see that $A = 0$, $B = -\frac{1}{2}$ satisfy this identity.

$\therefore$ the $n^{\text{th}}$ term of the series

$$= \frac{1}{n(n+1)(n+2)} - \frac{1}{2}\cdot\frac{1}{(n+1)(n+2)}\cdot\frac{1}{3^{n-1}} + \frac{1}{2}\cdot\frac{1}{n(n+1)}\cdot\frac{1}{3^{n-2}},$$

$\therefore$ the sum of $n$ terms

$$= C - \frac{1}{2(n+1)(n+2)} - \frac{1}{2}\cdot\frac{1}{(n+1)(n+2)}\cdot\frac{1}{3^{n-1}}.$$

To determine $C$, we observe that when $n = 0$, the sum $= 0$,

$$\therefore C = \frac{1}{2.1.2} + \frac{1}{2}\cdot\frac{1}{1.2}\cdot 3 = \frac{4}{4} = 1.$$

$$\therefore \text{sum of } n \text{ terms} = 1 - \frac{1}{2(n+1)(n+2)}\left(1 + \frac{1}{3^{n-1}}\right).$$

2. $m^{n-1} - 1 =$ mult. of $n$; $n^{m-1} - 1 =$ mult. of $m$.

$$\therefore (m^{n-1}-1)(n^{m-1}-1) = \text{mult. of } mn.$$

$$\therefore \left\{m^{n-1}.n^{m-1} + 1 - (m^{n-1} + n^{m-1})\right\} \div mn = \text{an integer}.$$

$$\therefore (m^{n-1} + n^{m-1} - 1) \div mn = \text{an integer}.$$

$\therefore m^{n-1} + n^{m-1}$ when divided by $mn$ leaves a remainder 1.

3. Since the ships are moving uniformly, we may imagine the first to remain at rest, and the other to be moving with a velocity equal to the difference between its own and that of the first. Let $A$ denote this stationary position of the first ship, $B$, $C$, $D$ the positions of the second at the different observations.

Then $\frac{p}{AC} = \frac{BC}{AC} = \frac{\sin(a-\beta)}{\sin(\theta-a)}$; $\frac{AC}{q} = \frac{AC}{CD} = \frac{\sin(\theta-\gamma)}{\sin(\beta-\gamma)}$,

$$\therefore \frac{p}{q} = \frac{\sin(a-\beta)\sin(\theta-\gamma)}{\sin(\beta-\gamma)\sin(\theta-a)}.$$

$\therefore p\sin(\beta-\gamma)(\sin\theta\cos\alpha-\cos\theta\sin\alpha) = q\sin(\alpha-\beta)(\sin\theta\cos\gamma-\cos\theta\sin\gamma)$,

$$\therefore \tan\theta\left\{p\cos\alpha\sin(\beta-\gamma) - q\cos\gamma\sin(\alpha-\beta)\right\}$$
$$= p\sin\alpha\sin(\beta-\gamma) - q\sin\gamma\sin(\alpha-\beta).$$

4. The angle $EBC = EDC = FDA = FBA$, and $AFB = BEC$,

$\therefore BEC = BAF$, and the triangles $ABF$, $CBE$ are similar.

$\therefore$ the angle $BCE = BAF = CDB = BEC$,

$\therefore$ the triangle $BEC$ is isosceles, and $\therefore$ also $AFB$.

5. In the ellipse $CG : CN :: CS^2 : CA^2$

$$Cg : PN :: CS^2 : CB^2,$$

$\therefore CG \cdot Cg \propto CN \cdot PN$. Now $PN^2 : AC^2 - CN^2 :: BC^2 : CA^2$,

$$PN^2 \cdot CN^2 = \frac{BC^2}{AC^2}(AC^2 - CN^2)\cdot CN^2$$
$$\propto (AC^2 - CN^2)\cdot CN^2.$$

Now the sum of these two factors is const. $\therefore$ the product is a max. when

$$CN^2 = AC^2 - CN^2, \quad \therefore CN^2 = \tfrac{1}{2}AC^2, \quad \therefore PN^2 = \tfrac{1}{2}BC^2,$$
$$\therefore CP^2 = CN^2 + PN^2 = \tfrac{1}{2}(AC^2 + BC^2),$$

$\therefore CP$ is one of the equiconjugate diamcters, and $\therefore$ 4 triangles can be drawn.

6. Let $ABC$ be the triangle, $O$ the circum-centre, $P$ the orthocentre. Let $D$, $D'$ be the middle points of $BC$, $AP$. Then if $DD'$ and $OP$ intersect in $N$, $DD'$ is the diameter and $N$ the centre of the 9 point circle of $ABC$, and $N$ bisects both $OP$ and $DD'$. $\therefore ODPD'$ is a parallelogram.

Then resultant of $PB$, $PC$ is represented by $2PD$; and $PA = 2PD'$,

$\therefore$ resultant of $PA$, $PB$, $PC$ = twice resultant of $PD$, $PD'$

$$= 2OP.$$

7. Let $(h, k)$ be the point from which the tangents are drawn.

Then by Paper XXXV., No. 7,

$$(\text{length of chord})^2 = 4\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right)\frac{\dfrac{b^2h^2}{a^2} + \dfrac{a^2k^2}{b^2}}{\left(\dfrac{h^2}{a^2} + \dfrac{k^2}{b^2}\right)^2}$$
$$= 4p^2, \text{ since } (h, k) \text{ is on the given curve.}$$

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## Paper LV.

1. Let $x, y, z$ denote the digits, $r$ the radix of the scale. Then the numbers are $r^2x + ry + z$, $r^2y + rz + x$, $r^2z + rx + y$. Since these are in A.P.

$$r^2(x + z) + r(x + y) + y + z = 2(r^2y + rz + x),$$

$$\therefore x(r^2 + r - 2) + y(-2r^2 + r + 1) + z(r^2 - 2r + 1) = 0,$$

$$\therefore x(r + 2)(r - 1) - y(2r + 1)(r - 1) + z(r - 1)^2 = 0, \text{ and } r - 1 \neq 0,$$

$$\therefore x(r + 2) - y(2r + 1) + z(r - 1) = 0,$$

$$\therefore z + x - 2y = \frac{3}{r - 1}(y - x) = t \text{ suppose.}$$

Then since $y - x < r - 1$, $\therefore \frac{y - x}{r - 1}$ is a positive or negative proper fraction. $\therefore$ $t$ is integral and lies between $+3$ and $-3$, and $\therefore = \pm 1$ or $\pm 2$.

$$\therefore 3(y - x) = (r - 1)t, \quad \therefore r = 1 + 3 \cdot \frac{y - x}{t} = 1 + 3p.$$

$\therefore$ the radix of the scale exceeds by unity a multiple of 3.

Now $y - x = pt$, and $z + x - 2y = t$, $\therefore z = x + (2p + 1)t$.

Now $z$ and $x$ are positive, and each $< 3p + 1$, $\therefore t \neq \pm 2$.

$$\therefore t = \pm 1.$$

If $t = +1$, $x$ may have any value between 1 and $p - 1$ inclusive.

" $t = -1$, $x$ " " " " $2p + 2$ and $3p$ "

$\therefore$ there are $2(p - 1)$ solutions.

Let any set of digits be $y = a$, $x = a + pt$, $z = a - (p + 1)t$, where $t = \pm 1$.

The com. dif. $= (3p+1)^2x + (3p+1)y + z - (3p+1)^2y - (3p+1)z - x$

$= (3p + 1)^2(x - y) + (3p + 1)(y - z) + z - x$

$= (3p + 1)^2 \cdot pt + (3p + 1)(p + 1)t - (2p + 1)t$

$= \pm 3p(3p^2 + 3p + 1)$

= constant, since $p$ is constant.

2. Denote the left hand member of the last line of the question by $\Sigma$. Multiply the first 3 equations by $\lambda_1$, $\mu_1$, $\nu_1$ respectively.

$$\therefore x^2\lambda_1 a_1 - xy\lambda_1 a_2 = \rho(\lambda_1\mu_2 - \lambda_1\nu_2),$$
$$x^2\mu_1 b_1 - xy\mu_1 b_2 = \rho(\mu_1\nu_2 - \mu_1\lambda_2),$$
$$x^2\nu_1 c_1 - xy\nu_1 c_2 = \rho(\nu_1\lambda_2 - \nu_1\mu_2).$$

$\therefore$ adding, we get

$$\rho\Sigma = x^2\sigma - xy(\lambda_1 a_2 + \mu_1 b_2 + \nu_1 c_2)$$
$$= \sigma(\rho^2 + y^2) - xy(\lambda_1 a_2 + \mu_1 b_2 + \nu_1 c_2)$$
$$= \sigma\rho^2 + \frac{y}{x}\left\{\sigma xy - x^2(\lambda_1 a_2 + \mu_1 b_2 + \nu_1 c_2)\right\}.$$

Now
$$a_2 x^2 - a_1 xy = \rho(\mu_1 - \nu_1),$$
$$b_2 x^2 - b_1 xy = \rho(\nu_1 - \lambda_1),$$
$$c_2 x^2 - c_1 xy = \rho(\lambda_1 - \mu_1).$$
$$\therefore x^2(\lambda_1 a_2 + \mu_1 b_2 + \nu_1 c_2) - \sigma xy$$
$$= \rho(\lambda_1\mu_1 - \lambda_1\nu_1 + \mu_1\nu_1 - \mu_1\lambda_1 + \nu_1\lambda_1 - \nu_1\mu_1) = 0,$$
$$\therefore \rho\Sigma = \rho^2\sigma, \quad \therefore \Sigma = \rho\sigma.$$

3. Tripos 1875. Monday afternoon. No. 12.

4. Let $O$ be the centre, and $Q$ the middle point of the arc $BC$. Join $AQ$ meeting $CP$ in $F$, and let $AO$ produced bisect the arc $CD$ in $G$.

Then the arcs $AP$ and $CQ$ together = the arc $PQ$.

The angle $PAF$ = sum of angles subtended at the circumference by the arcs $AP$ and $CQ$ = the angle $PFA$. $\therefore PA = PF$.

Again, since the arc $AP$ = $CG$, $\therefore$ the angle $PCA$ = $CAG$.

$\therefore PC$ is parallel to $AO$. Similarly it may be shewn that $AQ$ is parallel to $CO$. $\therefore AOCF$ is a parallelogram, and $CF = AO$.

$$\therefore PC = PF + FC = AP + AO.$$

This proves that $2\sin 54° = 1 + 2\sin 18°$.

5. Tripos 1875. Wednesday morning. No. 1.

6. Let the equations of the ellipse, director circle and tangent be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,;\; x^2 + y^2 = a^2 + b^2\,;\; \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$

The equation representing $CP$ and $CQ$ is evidently

$$x^2 + y^2 = (a^2 + b^2)\left(\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right)^2,$$

which on reduction becomes

$$y^2 + \frac{2b\sin\theta\cos\theta}{a^2\sin^2\theta - b^2\cos^2\theta}\,xy - \frac{b^2}{a^2}\,x^2 = 0.$$

Writing this in the form

$$(y - m_1x)(y - m_2x) = 0,$$

we see that $m_1m_2 = -\dfrac{b^2}{a^2}$, which is the condition that the lines should be conjugate diameters.

7. Suppose one of the distances, $AP$, to remain constant, *i.e.* suppose $P$ to move along the circumference of a circle centre $A$. Then we have to find the point $P$ on this circle for which $BP + PC$ is a min. Then $P$ is such that $BP$ and $CP$ make equal angles with the tangent at $P$, and $\therefore$ also with the normal $AP$. (See *Theory of Maximum and Minimum* by present writer). $\therefore$ the angle $APB = APC$. Similarly, by supposing $PB$ to remain constant whilst $PA$ and $PC$ vary we should find the angle $APB = BPC$. $\therefore$ when $AP$, $BP$ and $CP$ all vary, the point $P$ is such that $APB = BPC = CPA = 120°$. $\therefore$ on the sides describe arcs of circles containing 120°. The point of intersection of these arcs is the point required.

NOTE.—If one angle $C$ be greater than 120°, the point $P$ will evidently coincide with $C$.

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## PAPER LVI.

1. (1)

$$x^4y^3 + x^3y^4 = 27\,(2x^2y^2 - x - y),$$
$$x^2y + xy^2 = 3\,(4xy - x - y),$$
$$\therefore (x + y)(x^3y^3 + 27) = 54x^2y^2,$$
$$(x + y)(xy + 3) = 12xy,$$

$$\therefore \text{ dividing, } x^2y^2 - 3xy + 9 = \tfrac{9}{2}xy,$$

$$\therefore 2x^2y^2 - 15xy + 18 = 0,$$

$$\therefore (2xy - 3)(xy - 6) = 0, \quad \therefore xy = \tfrac{3}{2} \text{ or } 6.$$

| either $(x+y)(6+3) = 12 \times 6$ | or $(x+y)(\tfrac{3}{2}+3) = 12 \times \tfrac{3}{2}$ |
|---|---|
| $\therefore x + y = 8$, $xy = 6$ | $\therefore x + y = 4$, $xy = \tfrac{3}{2}$ |
| $\therefore x - y = \pm 2\sqrt{10}$ | $\therefore x - y = \pm \sqrt{10}$ |

$\therefore x = 4 \pm \sqrt{10}$; $y = 4 \mp \sqrt{10}$; or $x = \frac{1}{2}(4 \pm \sqrt{10})$; $y = \frac{1}{2}(4 \mp \sqrt{10})$.

Also $x = 0$, $y = 0$ obviously satisfy the equations.

(2) Multiply up, and we get

$$b^2z + c^2y = xyz = c^2x + a^2x = a^2y + b^2x,$$

$$\therefore (b^2 - c^2)x + a^2y - a^2z = 0$$

$$c^2x - c^2y + (a^2 - b^2)z = 0,$$

$$\therefore \frac{x}{a^2(b^2+c^2-a^2)} = \frac{y}{b^2(c^2+a^2-b^2)} = \frac{z}{c^2(a^2+b^2-c^2)} = R \text{ suppose.}$$

Substitute in $\dfrac{b^2}{y} + \dfrac{c^2}{z} = x,$

$$\therefore R^2(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2) = 2.$$

This determines $R$, and then $x$, $y$, $z$ can be found.

(3) $$(x^2 - a^2)\sqrt{4a^2 - x^2} = 2a^3,$$

$$\therefore (x^4 - 2a^2x^2 + a^4)(4a^2 - x^2) = 4a^6,$$

$$\therefore -x^6 + 6a^2x^4 - 9a^4x^2 + 4a^6 = 4a^6,$$

$$\therefore x^6 - 6a^2x^4 + 9a^4x^2 = 0,$$

$$\therefore x^2(x^2 - 3a^2)^2 = 0, \quad \therefore x = 0 \text{ or } \pm a\sqrt{3}.$$

2. $$1 + \sqrt{-3} = 2\left(\frac{1}{2} + \sqrt{-1}\cdot\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3} + \sqrt{-1}\sin\frac{\pi}{3}\right)$$

$$\therefore 4(1+\sqrt{-3})^{6n-1} = 4 \cdot 2^{6n-1}\left(\cos\frac{\pi}{3} + \sqrt{-1}\sin\frac{\pi}{3}\right)^{6n-1}$$

$$= 2^{6n+1}\left\{\cos(6n-1)\frac{\pi}{3} + \sqrt{-1}\sin(6n-1)\frac{\pi}{3}\right\}$$

$$= 2^{6n+1}\left(\cos\frac{\pi}{3} - \sqrt{-1}\sin\frac{\pi}{3}\right),$$

$$(1 - \sqrt{-3})^{6n+1} = 2^{6n+1}\left(\cos\frac{\pi}{3} - \sqrt{-1}\sin\frac{\pi}{3}\right)^{6n+1}$$

$$= 2^{6n+1}\left\{\cos(6n+1)\frac{\pi}{3} - \sqrt{-1}\sin(6n+1)\frac{\pi}{3}\right\}$$

$$= 2^{6n+1}\left(\cos\frac{\pi}{3} - \sqrt{-1}\sin\frac{\pi}{3}\right),$$

$$\therefore 4(1 + \sqrt{-3})^{6n-1} - (1 - \sqrt{-3})^{6n+1} = 0.$$

3. $\dfrac{7\sqrt{3} - 10}{\sqrt{3}} = \tan^2 x + \sec 2x$

$$= \frac{1 - \cos 2x}{1 + \cos 2x} + \sec 2x = \frac{\sec 2x - 1}{\sec 2x + 1} + \sec 2x$$

$$= 1 + \sec 2x - \frac{2}{1 + \sec 2x}.$$

$$\therefore (1 + \sec 2x)^2 - \frac{7\sqrt{3} - 10}{\sqrt{3}}(1 + \sec 2x) - 2 = 0,$$

$$\therefore 1 + \sec 2x = \frac{7\sqrt{3} - 10}{2\sqrt{3}} \pm \frac{\sqrt{271 - 140\sqrt{3}}}{2\sqrt{3}} = \frac{7\sqrt{3} - 10}{2\sqrt{3}} \pm \frac{14 - 5\sqrt{3}}{2\sqrt{3}},$$

$$\therefore \sec 2x = \frac{2}{\sqrt{3}} \text{ or } 5 - 4\sqrt{3}.$$

Now $\dfrac{7\sqrt{3} - 10}{\sqrt{3}}$ is negative, and $\tan^2 x$ is positive. $\therefore$ $\sec 2x$ is negative. $\therefore$ we must take the second value.

$$\therefore \cos 2x = \frac{1}{5 - 4\sqrt{3}} = -\frac{5 + 4\sqrt{3}}{23}.$$

*4. Let $A_1, A_2 \ldots A_7$ be the angular points of a regular heptagon inscribed in a circle, and let $O$ be any point on the circumference between $A_1$ and $A_7$.

Let $A_1A_2 = A_2A_3 = \ldots = a$; $A_1A_3 = A_2A_4 = \ldots = d$.

Then from the quadrilateral $OA_1A_2A_3$ we have by Euc. VI., D,

$$(OA_1 + OA_3)\,a = OA_2\,d; \quad \text{So } (OA_2 + OA_4)\,a = OA_3\,d;$$
$$(OA_3 + OA_5)\,a = OA_4\,d; \qquad (OA_4 + OA_6)\,a = OA_5\,d;$$
$$(OA_5 + OA_7)\,a = OA_6\,d; \qquad (OA_6 - OA_1)\,a = OA_7\,d;$$
$$(OA_2 - OA_7)\,a = OA_1\,d.$$

$\therefore$ by addition, we have

$$\{OA_2 + OA_4 + OA_6 - (OA_1 + OA_3 + OA_5 + OA_7)\}\,d$$
$$= 2a\,\{OA_1 + OA_3 + OA_5 + OA_7 - (OA_2 + OA_4 + OA_6)\},$$
$$\therefore \{OA_2 + OA_4 + OA_6 - (OA_1 + OA_3 + OA_5 + OA_7)\}\,(d + 2a) = 0.$$

Now $$d + 2a \neq 0,$$

$$\therefore OA_1 + OA_3 + OA_5 + OA_7 = OA_2 + OA_4 + OA_6.$$

The above method is perfectly general, and may be extended to the case of any regular figure of an odd number of sides.

5. Let $O$ be the fixed point, $AC$, $BD$ the chords through $O$ at right angles.

Let $E$, $F$, $G$, $H$ be the poles of $AB$, $BC$, $CD$, $DA$. Let $FG$, $GH$ meet in $K$, and let $EF$, $DG$ meet in $L$.

Then $AEB = \pi - 2EAB = \pi - 2ADB = 2DAC = 2CDG = \pi - CGD.$

$\therefore AEB + CGD = 2$ right angles.

$\therefore$ a circle can be described about $EFGH$.

Again, $F$ is the pole of $BC$, and $H$ is the pole of $AD$. $\therefore$ $FH$ is the polar of the point of intersection of $AD$ and $BC$, which evidently passes through $O$. Similarly $EG$ passes through $O$.

Again, $BEO = DHO$, being in the same segment,

and $EBO = \pi - LBO = \pi - LDO = HDO,$

$\therefore EOB = HOD = FOB.$ $\therefore$ $EOF$ is bisected by $OB$, and $\therefore$ also $EOH$ is bisected by $OA$.

6. $SP^2 + S'P^2 = 2(SO^2 + OP^2) = 2(SO^2 + BQ^2) = 2(SO^2 + OB^2 + OQ^2)$
$= 2\,(SB^2 + OQ^2) = 2\,(OA^2 + OQ^2) = AQ^2 + A'Q^2,$

and $SP + S'P = AA' = AQ + A'Q,$ $\therefore SP = AQ;$ $S'P = A'Q.$

7. Through $M$ draw $MH$ at right angles to $MN$, meeting the xis in $H$.

Then $HA \,.\, AN = AM^2 = PN^2 = 4AS \,.\, AN$. $\therefore\ AH = 4AS$. $\therefore$ $H$ is a fixed point, and $AM$ is a fixed line, and $MN$ at right angles to $MH$. $\therefore$ the envelope of $MN$ is a parabola having $H$ for focus and $AM$ for tangent at the vertex.

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## Paper LVII.

1. (1)
$$\begin{aligned} S &= 6 + 7 + 18 + 45 + 94 + 171 + \ldots \\ S_1 &= \quad 1 + 11 + 27 + 49 + 77 + \ldots \\ S_2 &= \quad\quad 10 + 16 + 22 + 28 + \ldots \\ S_3 &= \quad\quad\quad 6 + 6 + 6 + \ldots \end{aligned}$$

Here the 3rd difference series gives us a series of equal terms. $\therefore$ the $n^{\text{th}}$ term is of the 3rd degree in $n$, and is of the form

$$A + Bn + Cn\,(n+1) + Dn\,(n+1)\,(n+2).$$

Putting $n = 1, 2, 3, 4$ in succession, we have

$$\left.\begin{aligned} A + B + 2C + 6D &= 6 \\ A + 2B + 6C + 24D &= 7 \\ A + 3B + 12C + 60D &= 18 \\ A + 4B + 20C + 120D &= 45 \end{aligned}\right\}$$

From these equations, we get $A = 9$, $B = -1$, $C = -4$, $D = 1$.

$\therefore$ the $n^{\text{th}}$ term is $n\,(n+1)\,(n+2) - 4n\,(n+1) - n + 9$.

$\therefore$ the sum of $n$ terms

$$= \tfrac{1}{4}n\,(n+1)\,(n+2)\,(n+3) - \tfrac{4}{3}n\,(n+1)\,(n+2) - \tfrac{1}{2}n\,(n+1) + 9n$$

$$= \frac{n}{12}\left\{n\,(n+3)\,(3n-7) + 88\right\}.$$

(2)
$$\begin{aligned} S &= 7 + 13 - 7 - 181 - 1149 - 6111 - \ldots \\ S_1 &= \quad 6 - 20 - 174 - 968 - 4962 - \ldots \\ S_2 &= \quad\quad - 26 - 154 - 794 - 3994 - \ldots \\ S_3 &= \quad\quad\quad - 128 - 640 - 3200 - \ldots \end{aligned}$$

Here the 3rd difference series is a G.P. common ratio 5.

$\therefore$ the $n^{\text{th}}$ term is of the form $A + Bn + Cn(n+1) + D \cdot 5^{n-1}$.

Determine $A$, $B$, $C$, $D$ as before by giving $n$ the successive values 1, 2, 3, 4, and we find $A = 1$, $B = 2$, $C = 3$, $D = -2$.

$\therefore$ the $n^{\text{th}}$ term is $1 + 2n + 3n(n+1) - 2 \cdot 5^{n-1}$.

$\therefore$ the sum of $n$ terms $= n + n(n+1) + n(n+1)(n+2) - \frac{1}{2}(5^n - 1)$

$$= n(n+2)^2 - \tfrac{1}{2}(5^n - 1).$$

(3) By the method of III., p. 20, we find the $n^{\text{th}}$ term of the series formed by the numerators is

$$3 \cdot 2^{n-1} + n^2 - 12.$$

$\therefore$ the $n^{\text{th}}$ term of the given series

$$= \frac{3 \cdot 2^{n-1} + n^2 - 12}{n(n+1)(n+2)} \cdot \frac{1}{2^{n-1}}$$

$$= \frac{3}{n(n+1)(n+2)} + \frac{n^2 - 12}{n(n+1)(n+2)} \cdot \frac{1}{2^{n-1}}$$

As in V., p. 23, assume

$$\frac{n^2 - 12}{n(n+1)(n+2)} \cdot \frac{1}{2^{n-1}} \equiv \frac{An + B}{(n+1)(n+2)} \cdot \frac{1}{2^{n-1}} - \frac{A(n-1) + B}{n(n+1)} \cdot \frac{1}{2^{n-2}},$$

$$\therefore n^2 - 12 \equiv -An^2 - (2A + B)n + 4(A - B).$$

$\therefore$ equating coefficients of like powers of $n$, we find that $A = -1$, $B = 2$ satisfy this identity.

$\therefore$ the $n^{\text{th}}$ term $= \dfrac{(n-1) - 2}{n(n+1)} \cdot \dfrac{1}{2^{n-2}} - \dfrac{n-2}{(n+1)(n+2)} \cdot \dfrac{1}{2^{n-1}}$

$$= U_{n-1} - U_n.$$

$\therefore$ the sum of $n$ terms of the given series

$$= C - \frac{3}{2} \cdot \frac{1}{(n+1)(n+2)} - \frac{n-2}{(n+1)(n+2)} \cdot \frac{1}{2^{n-1}}.$$

If $n = 0$, the sum $= 0$, $\therefore C = -\frac{5}{4}$.

2. Tripos 1875. Thursday morning. No. 1.

3. From (2) $\quad 2\cos^2\phi + \cos^2\theta = 1,$

and $\quad \sin^2\theta + \cos^2\theta = 1.$

$$\therefore 2\cos^2\phi = \sin^2\theta, \quad \therefore 2\cos\phi = \sqrt{2}\sin\theta.$$

From (1) $\sin\phi\,(2\cos\phi - 1) = 1 - 2\cos\phi,$

$\therefore\ (\sin\phi + 1)(2\cos\phi - 1) = 0.$

If $\left.\begin{aligned} &\sin\phi + 1 = 0, \quad \phi = -\frac{\pi}{2}, \quad \theta = 0. \\ &2\cos\phi - 1 = 0, \quad \phi = \frac{\pi}{3}, \qquad \theta = \frac{\pi}{4}. \end{aligned}\right\}$

From these results, the general values of $\theta$, $\phi$ can be written down.

4. $(\sin A + \sin B)(\cos B + \cos C)(\cos C + \cos A)$

$= 8\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}\sin\frac{A+B}{2}\cos\frac{B+C}{2}\cos\frac{C+A}{2}$

$= 8\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}\cos\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2}.$

Now $\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2} + \sin\frac{C}{2}\sin\frac{A}{2}\cos\frac{B}{2}$

$= \sin\frac{B}{2}\cdot\sin\frac{A+C}{2} + \sin\frac{C}{2}\sin\frac{A}{2}\sin\frac{A+C}{2}$

$= \sin\frac{A+C}{2}\left(\cos\frac{A+C}{2} + \sin\frac{C}{2}\sin\frac{A}{2}\right)$

$= \sin\frac{A+C}{2}\cos\frac{A}{2}\cos\frac{C}{2}$

$= \sin\frac{A+C}{2}\sin\frac{B+C}{2}\sin\frac{A+B}{2},$

$\therefore$ the given expression

$= 8\sin\frac{A+B}{2}\cos\frac{A-B}{2}\cdot\sin\frac{B+C}{2}\cos\frac{B-C}{2}\cdot\sin\frac{C+A}{2}\cos\frac{C-A}{2}$

$= (\sin A + \sin B)(\sin B + \sin C)(\sin C + \sin A).$

5. Let $AD$ be the diameter of the circle, and let $ABCD$ be the figure formed by half the hexagon, so that $AB = a$, $BC = b$, $CD = c$. Join $AC$. Then $ACD$ is a right angle.

$\therefore\ d^3 = (AC^2 + CD^2)\,d$

$= (a^2 + b^2 - 2ab\cos B + c^2)\,d$

$= (a^2 + b^2 + c^2 + 2ab\cos D)\,d$, and $\cos D = \frac{c}{d}$,

$= (a^2 + b^2 + c^2)\,d + 2abc.$

Now the square of the area of the triangle, having $a\sqrt{2}$, $b\sqrt{2}$, $c\sqrt{2}$ for sides $= \frac{1}{4}(a+b+c)(a+b-c)(b+c-a)(c+a-b)$.

Square of area of hexagon $=$ 4 times the square of $ABCD$

$$= \tfrac{1}{4}(a+b+c-d)(a+b-c+d)(b+c-a+d)(c+a-b+d)$$

$$= \tfrac{1}{4}\left\{(a+b)^2-(c-d)^2\right\}\left\{(c+d)^2-(a-b)^2\right\}$$

$$= \tfrac{1}{4}\left\{(a+b)^2(c+d)^2+(a-b)^2(c-d)^2-(a^2-b^2)^2-(c^2-d^2)^2\right\}$$

$$= \tfrac{1}{4}\left\{2a^2c^2+2b^2c^2+2a^2b^2+2d^2(a^2+b^2+c^2)\right.$$
$$\left.-a^4-b^4-c^4-(a^2+b^2+c^2)d^2+6abcd\right\}$$

$$= \tfrac{1}{4}\left\{2a^2c^2+2b^2c^2+2a^2b^2-a^4-b^4-c^4+4abcd+d^4\right\}$$

$$= \tfrac{1}{4}(a+b+c)(a+b-c)(b+c-a)(c+a-b)+abcd+\tfrac{1}{4}d^4$$

$$= (\text{area of } \triangle)^2+abcd+\tfrac{1}{4}d^4.$$

6. Let $A$ be the centre of the fixed circle, $B$ the centre of the other, $C$ the fixed point, $DE$ the common chord. From $C$ draw $CG$ perpendicular to $EF$, and $CH$, a tangent to the fixed circle.

Then by Casey, p. 113, Cor. 1.

$$CH^2 = AB \,.\, CG.$$

Now $CH$ and $AB$ are fixed, $\therefore$ $CG$ is fixed.

7. Tripos 1878 Monday morning. No. 8.

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## Paper LVIII.

1. (1) $$\frac{x^2}{a}+\frac{y^2}{b}=\frac{a^2}{x}+\frac{b^2}{y}=a+b,$$

$$\therefore \frac{x^2-a^2}{a}=-\frac{y^2-b^2}{b}, \text{ and } \frac{a(x-a)}{x}=-\frac{b(y-b)}{y} \quad \ldots\ldots (1)$$

$$\therefore \text{ by division } \frac{x(x+a)}{a^2}=\frac{y(y+b)}{b^2},$$

$$\therefore b^2x^2-a^2y^2+ab^2x-a^2by=0,$$

$$\therefore (bx-ay)(bx+ay+ab)=0.$$

If $bx-ay=0$, then writing (1)

$$\frac{ab(x-a)}{bx}+\frac{ab(y-b)}{ay}=0,$$

we have $x + y = a + b$, $\therefore x = a, y = b$ . . . . . (*A*).

If $bx + ay + ab = 0$,

$$\therefore y = -\frac{(a+x)b}{a}, \qquad \therefore \frac{b^2}{y} = -\frac{ab}{a+x},$$

$$\therefore a + b = \frac{a^2}{x} + \frac{b^2}{y} = \frac{a^2}{x} - \frac{ab}{a+x},$$

$$\therefore (a+b)x^2 + 2abx - a^3 = 0,$$

$$\therefore x = \frac{-ab \pm a\sqrt{a^2 + ab + b^2}}{a+b}, \text{ or } a.$$

$$y = -\frac{(a+x)b}{a} = -1 - \frac{b}{a}x = \frac{-b^2 \mp \sqrt{a^2 + ab + b^2}}{a+b}, \text{ or } b.$$

(2) Write the equations in the homogeneous form

$$x^2(a+b) - bxy - ay^2 = a - c \quad (1).$$

$$bx^2 + axy - (a+b) = b - c \quad (2).$$

Eliminate the absolute terms by multiplying (1) by $b - c$, and (2) by $a - c$ and subtracting.

$$\therefore x^2(b^2 - ac) - xy(b^2 - ac + a^2 - bc) + y^2(a^2 - bc) = 0.$$

Divide by $y^2$, and for $\frac{x}{y}$ write $z$.

$$\therefore z^2(b^2 - ac) - z(b^2 - ac + a^2 - bc) + a^2 - bc = 0 \quad . \quad . \quad . \quad (3).$$

Evidently $z = 1$ satisfies this equation. This leads to the result $x = y$, which we see is inadmissible.

The product of the roots of (3) is $\frac{a^2 - bc}{b^2 - ac}$, and since one root is unity, the other is $\frac{a^2 - bc}{b^2 - ac}$. This leads to

$$\frac{x}{a^2 - bc} = \frac{y}{b^2 - ac} = R \text{ suppose} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Now subtract (2) from (1),

$$\therefore ax^2 - (a+b)xy + by^2 = a - b,$$

$$\therefore (x - y)(ax - by) = a - b.$$

Substitute in this equation the values of $x$ and $y$ from (4),

$$\therefore R^2(a^2 - b^2 + ac - bc)(a^3 - b^3) = a - b,$$

$$\therefore R(a + b + c)(a^3 - b^3) = 1$$

This gives the value of $R$, and $\therefore$ $x$ and $y$ are known.

2. $(e^x - 1)^n = \left(x + \frac{x^2}{\lfloor 2} + \ldots\right)^n = x^n +$ terms involving higher powers of $x$.

Now $(e^x - 1)^n = e^{nx} - ne^{(n-1)x} + \frac{n(n-1)}{\lfloor 2} e^{(n-2)x} - \ldots$ by the Binomial Theorem.

$\therefore$ expanding each of the terms on the right, we find the coef. of $x^3$ is

$$\frac{1}{\lfloor 3}\left\{n^3 - n(n-1)^3 + \frac{n(n-1)}{\lfloor 2} \cdot (n-2)^3 - \ldots\right\}.$$

But no power of $x$ lower than $x^n$ occurs in $(e^x - 1)^n$.

$$\therefore n^3 - n(n-1)^3 + \frac{n(n-1)}{\lfloor 2}(n-2)^3 - \ldots = 0 \ldots (A),$$

and by p. 6, Art. 7, of Problem Papers,

$$n^3 + n(n-1)^3 + \frac{n(n-1)}{\lfloor 2}(n-2)^3 + \ldots = n^2(n+3)2^{n-3} \ldots (B).$$

$\therefore$ adding $(A)$ and $(B)$, and dividing by 2 we get the required result.

3. Clear of fractions.

$$\sin(\alpha+\theta)\sin(\beta+\phi) + \sin(\alpha+\phi)\sin(\beta+\theta) = 2\sin(\alpha+\phi)\sin(\beta+\phi),$$

$$\cos(\alpha+\theta)\cos(\beta+\phi) + \cos(\alpha+\phi)\cos(\beta+\theta) = 2\cos(\alpha+\phi)\cos(\beta+\phi).$$

$\therefore$ adding these two lines together we have

$$\cos(\alpha - \beta + \theta - \phi) + \cos(\alpha - \beta - \theta + \phi) = 2\cos(\alpha - \beta),$$

$$\therefore 2\cos(\alpha - \beta)\cos(\theta - \phi) = 2\cos(\alpha - \beta),$$

$$\therefore \cos(\alpha - \beta) = 0, \quad \text{or } \cos(\theta - \phi) = 1,$$

$$\therefore \alpha \sim \beta = (2n+1)\frac{\pi}{2}, \text{ or } \theta \sim \phi = 2n\pi.$$

4. $\tan\theta > \theta$, $\therefore \tan\frac{1}{1+x^2} > \frac{1}{1+x^2}$ which is $> \frac{1}{1+x+x^2}$.

Now $\sin\theta < \theta$, and $1 - \frac{\theta^2}{2} < \cos\theta$,

$$\therefore \tan\theta < \frac{\theta}{1 - \frac{\theta^2}{2}} \quad \therefore \tan\frac{1}{1+x^2} < \frac{\frac{1}{1+x^2}}{1 - \frac{1}{2(1+x^2)^2}}$$

$$\therefore \tan\frac{1}{1+x^2} < \frac{1}{1-x+x^2}.$$

$$\text{if } \frac{\frac{1}{1+x^2}}{1 - \frac{1}{2(1+x^2)^2}} < \frac{1}{1-x+x^2};$$

$$\text{if } \frac{2(1+x^2)}{2(1+x^2)^2 - 1} < \frac{1}{1-x+x^2};$$

$$\text{if } 2(1 - x + 2x^2 - x^3 + x^4) < 2(1 + 2x^2 + x^4) - 1;$$

$$\text{if } 1 < 2x(1 + x^2)$$

which is the case if $x > \frac{1}{2}$.

5. Tripos 1878. Wednesday morning. No. 1.

*6. Let $y^2 = 4ax$ be the equation to the parabola, $(x_1y_1)$, $(x_2y_2)$ the coordinates of the extremities of the chord $PQ$. Then the equation to $PQ$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) = \frac{4a(y_2 - y_1)}{x_2 - x_1}(x - x_1)$$

$$= \frac{4a}{y_1 + y_2}(x - x_1);$$

or
$$y(y_1 + y_2) - 4ax = y_1y_2 \quad (A).$$

If $(\xi, \eta)$ be the coordinates of $R$, the middle point of $PQ$, and $y_1 - y_2 = 2c$, we have

$$2\xi = x_1 + x_2 = \frac{1}{4a}(y_1^2 + y_2^2); \; 2\eta = y_1 + y_2.$$

Now $(y_1 + y_2)^2 + (y_1 - y_2)^2 = 2(y_1^2 + y_2^2)$,

$$\therefore 4\eta^2 + 4c^2 = 16a\xi; \quad \therefore \eta^2 = 4a\xi - c^2. \quad (B).$$

To find the envelope of $PQ$, we have

$$(y_1 + y_2)^2 = (y_1 - y_2)^2 + 4y_1y_2 = 4c^2 + 4y_1y_2$$

$\therefore$ the equation $(A)$ to $PQ$ may be written

$$(y_1 + y_2)^2 - 4y(y_1 + y_2) + 16ax - 4c^2 = 0.$$

$\therefore$ expressing the condition that this equation should have equal roots, the equation of the envelope is

$y_2 = 4ax - c^2$, which coincides with $(B)$, the locus of $R$.

This question may also be solved geometrically.

Let $QVQ'$ be a double ordinate of the diameter $PV$, and $QD$ perpendicular to $PV$. Then $QD$ is constant, being half the difference of the ordinates of $Q$ and $Q'$, and $QD^2 = 4AS.PV$, $\therefore$ $PV$ is constant. Also $QVQ'$ is parallel to the tangent at $P$. $\therefore$ the locus of $V$, and the envelope of $QQ'$ coincide with the parabola obtained by moving the given parabola a constant distance $PV$ in the direction of its axis. See Errata.

7. Let $(a \cos \alpha, b \sin \alpha)$ be the coordinates of $P$. Then those of $p$ are $(a \cos \alpha, a \sin \alpha)$.

The equation of $CP$ is $y = \frac{b}{a} x \tan \alpha$.

,, auxiliary circle is $x^2 + y^2 = a^2$,

$\therefore$ by substitution $x^2 + \frac{b^2}{a^2} x^2 \tan^2 \alpha = a^2$.

$\therefore$ coordinates of $q$ are

$$x = \frac{a^2 \cos \alpha}{\sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}}, \quad y = \frac{ab \sin \alpha}{\sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}}$$

$\therefore$ coordinates of $Q$ are

$$x = \frac{a^2 \cos \alpha}{\sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}}, \quad y = \frac{b^2 \sin \alpha}{\sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}}.$$

$\therefore$ equation of $QT$, the tangent at $Q$ is

$$x \cos \alpha + y \sin \alpha = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}.$$

This is perpendicular to $Cp$, whose equation is

$$x \sin \alpha - y \cos \alpha = 0.$$

If $Cp$ and $QT$ intersect in $T$, $CT$ is the length of the perpendicular from $C$ on $QT$.

$$\therefore \quad CT = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha} = CP.$$

## Paper LIX.

1. By ordinary division, we find

$$\frac{1}{17} = \cdot\dot{0}588235294117647\dot{}.$$

The explanation required is exactly the same as that given in Papers XXVI., No. 1, XXXVI., No. 1.

2. (1) $$\frac{x-a}{b} - \frac{b}{x-a} = \frac{a}{x-b} - \frac{x-b}{a},$$

$$\therefore \frac{(x-a)^2 - b^2}{b(x-a)} = -\frac{(x-b)^2 - a^2}{a(x-b)},$$

$$\therefore \frac{(x-a-b)(x-a+b)}{b(x-a)} = -\frac{(x-b-a)(x-b+a)}{a(x-b)}$$

$$\therefore \text{either } x-a-b=0, \ \therefore x = a+b,$$

or $$a(x-b)(x-a+b) + b(x-a)(x-b+a) = 0.$$

$$\therefore x^2(a+b) - x(a^2+b^2) = 0,$$

$$\therefore x = 0, \ \frac{a^2+b^2}{a+b}, \ a+b.$$

(2) Square both sides, and transpose

$$\therefore \sqrt{x} + \sqrt{x-1} - \sqrt{x+1} = -2\sqrt{x^2-x}.$$

$$\therefore x + x - 1 + x + 1 + 2\sqrt{x^2-x} - 2\sqrt{x^2+x} - 2\sqrt{x^2-1} = 4\sqrt{x^2-x}.$$

$$\therefore 3x - 2\sqrt{x^2-1} = 2(\sqrt{x^2+x} + \sqrt{x^2-x}).$$

$$\therefore 9x^2 + 4(x^2-1) - 12x\sqrt{x^2-1} = 4(x^2+x+x^2-x+2x\sqrt{x^2-1}).$$

$$\therefore 5x^2 - 4 = 20x\sqrt{x^2-1}.$$

$$\therefore 25x^4 - 40x^2 + 16 = 400x^2(x^2-1).$$

$$\therefore 375x^4 - 360x^2 - 16 = 0.$$

$$\therefore 375x^2 = 180 \pm \sqrt{32400 + 6000}$$

$$= 180 \pm 80\sqrt{6}.$$

$$\therefore x^2 = \frac{4}{9 \times 25}(27 \pm 12\sqrt{6}).$$

$$\therefore x = \pm \frac{2}{15}\sqrt{27 \pm 12\sqrt{6}}.$$

3. $\cos A = \dfrac{(d-a)(b-c)}{(d+a)(b+c)} = \dfrac{bd + ac - ab - cd}{bd + ac + ab + cd}.$

$$\therefore \tan^2 \frac{A}{2} = \frac{1 - \cos A}{1 + \cos A} = \frac{ab + cd}{ac + bd}.$$

Similarly

$$\tan^2 \frac{B}{2} = \frac{bc + ad}{ba + cd}, \quad \tan^2 \frac{C}{2} = \frac{ca + bd}{cb + ad}.$$

Now $\dfrac{A}{2} + \dfrac{B}{2} + \dfrac{C}{2} = \pi,$

$$\therefore 0 = \tan\left(\frac{A}{2} + \frac{B}{2} + \frac{C}{2}\right)$$

$$= \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} - \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2},$$

$$\therefore \left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right)^2 = \tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan^2\frac{C}{2}$$

$$= \frac{ab + cd}{ac + bd} \cdot \frac{bc + ad}{ba + cd} \cdot \frac{ca + bd}{cb + ad}$$

$$= 1.$$

$$\therefore \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \pm 1.$$

4. Let $AR$ produced meet $CQ$ in $D$ and $BP$ in $E$, and let $BP$ and $CQ$ meet in $F$. Let $AQ$ produced meet $PC$ in $G$ and $BR$ in $H$, and let $BR$ and $CP$ meet in $K$.

Then the angle $EDF = ERB = ACB,$

and the angle $EFD = HQD = ABC,$

$\therefore$ the triangles $DEF$, $ABC$ are similar.

Also the angle $GHK = DFE$, and $GKH = EDF$, for $HB$ is parallel to $DF$, and $HG$ to $EF$, $\therefore$ the triangle $GHK$ is similar to $DEF$, and $\therefore$ also to $ABC$, and the triangles $GHK$, $DEF$ evidently have their homologous sides parallel.

Let the angle $ABE$ be denoted by $\theta$.

Then the sum of the homologous sides of $DEF$ and $GHK$ : the homologous side of $ABC$ :: $DF + HK : BC$.

Now $\frac{HK}{BC} = \frac{HB}{AB} \cdot \frac{AB}{BC} + \frac{BK}{BC} = \frac{\sin\theta}{\sin B} \cdot \frac{\sin C}{\sin A} + \frac{\sin(\theta + C)}{\sin C}$,

$$\frac{DF}{BC} = \frac{\sin\theta}{\sin C} \cdot \frac{\sin B}{\sin A} + \frac{\sin(\theta - B)}{\sin B}.$$

$$\therefore HK + DF : BC :: \sin\theta(\sin^2 C + \sin^2 B)$$

$$+ \{\sin(\theta + C)\sin B + \sin(\theta - B)\sin C\}\sin A : \sin A \sin B \sin C.$$

$$:: \sin\theta(\sin^2 A + \sin^2 B + \sin^2 C) : \sin A \sin B \sin C,$$

for $$\sin(\theta + C)\sin B + \sin(\theta - B)\sin C$$

$$= \sin B(\sin\theta\cos C + \cos\theta\sin C) + \sin C(\sin\theta\cos B - \cos\theta\sin B)$$

$$= \sin\theta\sin(B + C) = \sin\theta\sin A.$$

5. Tripos 1878. 2nd Tuesday morning. No. 1.

6. Tripos 1875. Wednesday morning. No. 8.

7. Tripos 1878. 2nd Monday afternoon. No. 4.

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## Paper LX.

1. Let $w, x, y, z$ be the number of Liberals returned by England, Scotland, Ireland, and Wales, respectively.

$W, X, Y, Z$ . . . . . Conservatives . . . . . .

$\therefore$ (1) $w + x + y + z = W + 15$; (2) $W + X + Y + Z = 2w + 15$;

(3) $X = z$; (4) $x - X = 2Z = \frac{2}{3}(y - Y)$. . . . (5).

(6) $W - w = Y + y + 10$;

(7) $W + X + Y + Z + w + x + y + z = 652$; (8) $x + X = 60$.

By (4) $x - X = 2Z$; by (8) $x + X = 60$.

$\therefore x = 30 + Z$; and $X = 30 - Z$. But $X = z$, $\therefore Z + z = 30$.

$$\therefore W + Y + w + y + 90 = 652,$$

$\therefore$ by (6) $$W + w + W - w + 80 = 652,$$

$\therefore 2W = 572.$ $\therefore W = 286$

From (1), (2), (8) $W + 2w = 632$, $\therefore 2w = 346.$ $\therefore w = 173$

From (2) $286 + z + Y + Z = 351$ $Y = 35$

$\therefore z + Y + Z = 65.$ $y = 68$

And $Z + z = 30$ $\therefore Y = 35$; and $\therefore$ by (6) $y = 68.$ $Z = 11$

From (5) $3Z = y - Y = 33$ $\therefore Z = 11$, and $Z + z = 30.$ $\therefore z = 19$

$X = z.$ $\therefore X = 19$

$X + x = 60.$ $\therefore x = 41$

2. Tripos 1878. 2nd Monday afternoon. No. 1.

3. $$\tan^2 \frac{C}{2} = \frac{(s-a)(s-b)}{s(s-c)},$$

$$\therefore \tan^2 \frac{B}{2} \tan^2 \frac{C}{2} = \frac{(s-a)(s-c)}{s(s-b)} \cdot \frac{(s-a)(s-b)}{s(s-c)} = \frac{(s-a)^2}{s^2},$$

$\therefore$ the given expression $= \dfrac{(s-a)^2 + (s-b)^2 + (s-c)^2}{s^2}.$

Now this fraction is $< 1$

if $(s-a)^2 + (s-b)^2 + (s-c)^2 < s^2,$

if $3s^2 - 4s^2 + a^2 + b^2 + c^2 < s^2,$

if $2(a^2 + b^2 + c^2) < (a+b+c)^2,$

if $a^2 + b^2 + c^2 < 2(bc + ca + ab).$

Now $a < b + c,$ $\therefore a^2 < a(b+c)$; $b^2 < b(c+a)$;

$c^2 < c(a+b)$; $\therefore a^2 + b^2 + c^2 < 2(bc + ca + ab).$

If $C$ is nearly $= 2$ right angles, $a + b = c,$ $\therefore s = c,$

$$\therefore \text{Expression} = \frac{1}{c^2}\left\{(c-a)^2 + (c-b)^2\right\}$$

$$= \frac{1}{c^2}\left\{2c^2 - 2c(a+b) + a^2 + b^2\right\} = \frac{a^2 + b^2}{c^2}.$$

Now $2(a^2 + b^2) = (a+b)^2 + (a-b)^2.$ $\therefore$ if $a + b$ is constant, and $a^2 + b^2$ is a min. $a = b = \dfrac{c}{2}.$ $\therefore$ given expression then $= \dfrac{1}{2}.$

4. Tripos 1878. Monday morning. No. 2.

5. $O$ is the centre of the inscribed circle, $\therefore$ if we draw $OD$, $OE$, $OF$, perpendicular to $BC$, $CA$, $AB$ respectively, $OD = OE = OF$.

$$\therefore \triangle BOC : \triangle COA : \triangle AOB :: \tfrac{1}{2} OD \,.\, BC : \tfrac{1}{2} OE \,.\, CA : \tfrac{1}{2} OF \,.\, AB$$
$$:: BC : CA : AB$$
$$:: \sin A : \sin B : \sin C.$$

6. Let $FF'$, $GG'$ be the intercepts on the circle intersecting in $T$. Join $CF'$, $CG'$, and draw $CN$, $CM$ perpendicular to $FF'$, $GG'$. Then $CNTM$ is a rectangle, and $\therefore MN^2 = CT^2 = a^2 + b^2$.

$$\text{Now } FF'^2 + GG'^2 = 4(F'N^2 + G'M^2) = 4\{2a^2 - (CN^2 + CM^2)\}$$
$$= 4(2a^2 - MN^2)$$
$$= 4\{2a^2 - (a^2 + b^2)\} = 4(a^2 - b^2) = \text{constant.}$$

7. Produce $PQ$ to meet the asymptotes in $P'$, $Q'$. Through $P$, $O$, $Q$ draw lines parallel to $CQ'$ and $CP'$, meeting $CP'$ in $N$, $K$, $D$, and $CQ'$ in $M$, $H$, $R$. Let $HO$ meet the lines through $P'$, $P$, $Q$ parallel to $CQ'$ in $E$, $F$, $X$, and let $OK$ meet $PL$ in $Y$.

Then $LM = QR = P'N = EF$. $\therefore LH = EP = \text{comp. } PK$.

$$\therefore LO = HK - (MK + LH) = HK - MN = \text{const.}$$
$$\therefore LX \,.\, LY = \text{const.}$$

$\therefore$ the locus of $L$ is a similar and similarly situated hyperbola, having its centre at $O$.

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## Paper LXI.

$$1. \quad \frac{1}{1-x} - \frac{1}{1-x^{2n+1}} = \frac{x - x^{2n+1}}{(1-x)(1-x^{2n+1})} = \frac{x(1-x^{2n})}{(1-x)(1-x^{2n+1})}$$

$$\frac{1}{1-x^3} - \frac{1}{1-x^{2n+3}} = \frac{x^3 - x^{2n+3}}{(1-x^3)(1-x^{2n+3})} = \frac{x^3(1-x^{2n})}{(1-x^3)(1-x^{2n+3})}$$

. . . . . .

$$\frac{1}{1-x^{2n+1}} - \frac{1}{1-x^{4n+1}} = \frac{x^{2n+1} - x^{4n+1}}{(1-x^{2n+1})(1-x^{4n+1})} = \frac{x^{2n+1}(1-x^{2n})}{(1-x^{2n+1})(1-x^{4n+1})}$$

. . . . . .

$\therefore$ the given series

$$= \frac{1}{1-x^{2n}}\left\{\frac{1}{1-x} + \frac{1}{1-x^3} + \ldots + \frac{1}{1-x^{2n+1}} + \ldots \right.$$
$$\left. - \frac{1}{1-x^{2n+1}} - \ldots \right\}$$
$$= \frac{1}{1-x^{2n}}\left\{\frac{1}{1-x} + \frac{1}{1-x^3} + \ldots + \frac{1}{1-x^{2n-1}}\right\}.$$

2. $\cos A \cos B + \cos C = \cos A \cos B - \cos(A+B) = \sin A \sin B.$

$$\sqrt{y^2+z^2+2yz\cos A} + \sqrt{z^2+x^2+2zx\cos B} = -\sqrt{x^2+y^2+2xy\cos C},$$

$$\therefore y^2+z^2+2yz\cos A + z^2+x^2+2zx\cos B - x^2-y^2-2xy\cos C$$
$$= -2\sqrt{(y^2+z^2+2yz\cos A)(z^2+x^2+2zx\cos B)}.$$

Divide by 2, square both sides and transpose.

$$\therefore y^2z^2(1-\cos^2 A) + 2xy^2z(\cos B + \cos A\cos C),$$
$$+ x^2y^2(1-\cos^2 B) + 2xyz^2(\cos C + \cos A\cos B),$$
$$+ x^2z^2(1-\cos^2 C) + 2x^2yz(\cos A + \cos B\cos C) = 0.$$
$$\therefore y^2z^2\sin^2 A + z^2x^2\sin^2 B + x^2y^2\sin^2 C$$
$$+ 2x^2yz\sin B\sin C + 2xy^2z\sin C\sin A + 2xyz^2\sin A\sin B = 0.$$
$$\therefore (yz\sin A + zx\sin B + xy\sin C)^2 = 0.$$

3. Let $ABCD$ be the quadrilateral. Produce $AB$, $DC$ to meet in $F$, and $DA$, $CB$ to meet in $E$. Let $AB = a$, $BC = b$, $CD = c$, $DA = a$.

Then $\dfrac{\sin(C+D)}{\sin C} = \cos D + \dfrac{\sin D}{\sin C}\cos C$

$$= \frac{c^2+d^2-a^2-b^2}{2(ac+bd)} + \frac{ac+bd}{ab+dc}\cdot\frac{b^2+c^2-d^2-a^2}{2(ac+bd)}.$$

Todh. *Trig.* Art 254.

$$= \frac{c^2-a^2}{ab+cd}.$$

$$\therefore ED = \frac{c\sin C}{\sin(C+D)} = \frac{c(ab+cd)}{c^2-a^2}.$$

Similarly $DF = \dfrac{d(ab + cd)}{d^2 - b^2}$.

$$\therefore\ EF^2 = DE^2 + DF^2 - 2DE \,.\, DF \cos D$$

$$= (ab + cd)^2 \left\{ \frac{c^2}{(c^2 - a^2)^2} + \frac{d^2}{(b^2 - d^2)^2} - \frac{cd(c^2 + d^2 - a^2 - b^2)}{(c^2 - a^2)(d^2 - b^2)(ab + cd)} \right\}$$

$$= \frac{(ab + cd)}{(c^2 - a^2)^2(b^2 - d^2)^2} \times N, \text{ where } N$$

$$= (ab+cd)\{c^2(b^2-d^2)^2+d^2(c^2-a^2)^2\} - cd(c^2-a^2)(d^2-b^2)(c^2+d^2-a^2-b^2)$$

$$= \{c^2(b^2-d^2)^2+d^2(c^2-a^2)^2\}(ab+cd-cd)+cd\{a^2(b^2-d^2)^2+b^2(c^2-a^2)^2\}$$

$$= (b^2 - d^2)^2(abc^2 + a^2cd) + (c^2 - a^2)(abd^2 + b^2cd)$$

$$= (ad + bc)\{bd(c^2 - a^2)^2 + ac(b^2 - d^2)^2\},$$

which is the required result.

4. Tripos 1878. Monday morning. No. 5.

5. Let $A$, $A'$ be the vertices of the hyperbola, $a$, $a'$ the corresponding vertices of the ellipse. Let $P$ be a point of intersection on the curve with vertex $A'$. Draw the tangent to the ellipse at $P$, and let the diameter parallel to this tangent meet $SP$, $PS'$ produced in $F$ and $F'$.

Then $PF = Ca = \frac{1}{2}(SP + S'P)$,

$$AC = \tfrac{1}{2}(SP - S'P),$$

$$\therefore\ AC + PF = SP,\ \ \therefore\ AC = SP - PF = SF.$$

$\therefore$ Locus of $F$ is a circle, centre $S$, and radius $CA$.

Similarly $PF' = Ca = \frac{1}{2}(SP + S'P)$

$$AC = \tfrac{1}{2}(SP - S'P).$$

$$\therefore\ PF' - AC = S'P,\ \ \therefore\ AC = PF' - S'P = F'S.$$

$\therefore$ Locus of $F'$ is a circle, centre $S'$, and radius $CA$.

*6. Bisect $PD$ at $U$, and $P'D'$ at $U'$. Draw the ordinates $P'PM$, $D'DN$. Let $UU'$ meet the axis in $L$. Then $UU'$ is parallel to $PP'$ and $DD'$, and is $\therefore$ perpendicular to the axis.

Then $\tan ACU : \tan ACU' :: \frac{UL}{CL} : \frac{U'L}{CL}$

$:: UL : U'L$

$:: PM + DN : P'M + D'N$

$:: b : a.$ See Errata.

7. Take the centre of the given circle as origin, and the axis of $y$ parallel to the given straight line. Then the equations of the circle, line, and point may be written

$$x^2 + y^2 = c^2;\ y = k;\ (x - \xi)^2 + (y - k)^2 = 0.$$

$\therefore$ the equation of the radical axis of the point and circle is

$$2x\xi + 2yk = c^2 + \xi^2 + k^2,$$

$$\therefore \xi^2 - 2x\xi + c^2 + k^2 - 2yk = 0.$$

$\therefore$ Expressing the condition that $\xi$ may have equal roots, we obtain as the equation of the envelope, $x^2 = c^2 + k^2 - 2yk$, which represents a parabola.

*This question may also be solved geometrically as follows. Let $O$ be the centre of the given circle, $PD$ the fixed straight line. Draw the radius $OAD$ at right angles to $PD$, and in $PD$ take any point $P$.

In $OD$ take a point $E$ such that $OE \cdot OD = OA^2$, and in $OP$ take $R$ such that $OR \cdot OP = OA^2$. Then $RE$ is the polar of $P$ with respect to the circle, and is perpendicular to $OP$. Bisect $PR$ in $Q$, and draw $VQV'$ at right angles to $OP$ meeting $EP$ in $Y$. Then since $OQ^2 - PQ^2 = OP.OR = (\text{radius})^2$

$\therefore$ $VQV'$ is the radical axis of the point circle at $P$ and the given circle.

Since $RE$ and $QV$ are parallel, both being perpendicular to $OP$, and $PQ = QR$, $\therefore$ if $EP$ meet $QV$ in $Y$, $EY = YP = YD$, since $EDP$ is a right angle. Also, if $S$ be the middle point of $OE$, $SY$ is parallel to $OP$ and perpendicular to $VQV'$.

Now since $YD = YP =$ tangent from $Y$ to the circle, $\therefore$ $Y$ lies on a fixed straight line $YX$ bisecting $ED$ at right angles.

$\therefore$ the envelope of $VQV'$ is a parabola of which $S$ is the focus, $OX$ the axis, and $YX$ the tangent at the vertex.

## Paper LXII.

1. (1) The $n^{th}$ term is $\dfrac{3 \cdot 6 \ldots 3n}{1 \cdot 4 \cdot 7 \ldots 3n+1}$.

This can be written $\dfrac{3 \cdot 6 \ldots 3n}{1 \cdot 4 \cdot 7 \ldots 3n+1} \left\{\dfrac{3n+3-(3n+1)}{2}\right\}$

$$= \tfrac{1}{2}\left\{\frac{3 \cdot 6 \ldots 3n+3}{1 \cdot 4 \cdot 7 \ldots 3n+1} - \frac{3 \cdot 6 \ldots 3n}{1 \cdot 4 \cdot 7 \ldots 3n-2}\right\},$$

which is of the form $\frac{1}{2}(U_n - U_{n-1})$.

The first term is $\frac{1}{2}\left(\frac{3 \cdot 6}{1 \cdot 4} - \frac{3}{1}\right)$,

$\therefore$ the sum of $n$ terms $= \frac{1}{2}\left\{\dfrac{3 \cdot 6 \ldots 3n+3}{1 \cdot 4 \cdot 7 \ldots 3n+1} - 3\right\}$.

(2) The $n^{th}$ term $= \dfrac{n}{(2n-1)(2n+1)(2n+3)}$

$$= \frac{2n-1+1}{2(2n-1)(2n+1)(2n+3)}$$

$$= \tfrac{1}{2}\left\{\frac{1}{(2n+1)(2n+3)} + \frac{1}{(2n-1)(2n+1)(2n+3)}\right\}.$$

Here the factors in the denominator of each fraction form a series in A.P. of which the common difference is the same as the coefficient of $n$ in each factor, and we employ the following rule: In each fraction take away one factor from the beginning, and divide by the number of remaining factors multiplied by the common difference of the A.P. Write the result negatively, and add a constant $C$.

$\therefore$ the sum of $n$ terms of the given series

$$= C - \tfrac{1}{2}\left\{\frac{1}{2(2n+3)} + \frac{1}{4(2n+1)(2n+3)}\right\}.$$

To find the value of $C$, put $n = 0$, and express the fact that the sum of no terms is zero.

$$\therefore 0 = C - \tfrac{1}{2}\left(\tfrac{1}{6} + \tfrac{1}{12}\right). \quad \therefore C = \tfrac{1}{8}.$$

On reduction it will be found that the required sum

$$= \frac{n(n+1)}{2(2n+1)(2n+3)}.$$

(3) $S = -3 + 2 + 13 + 28 + 39 + 26 - 55 - 296\ldots$

$S_1 = 5 + 11 + 15 + 11 - 13 - 81 - 241\ldots$

$S_2 = 6 + 4 - 4 - 24 - 68 - 160\ldots$

$S_3 = -2 - 8 - 20 - 44 - 92\ldots$

$S_4 = -6 - 12 - 24 - 48\ldots$

Here the 4th difference series is a geometric progression, common ratio 2, $\therefore$ the $n^{\text{th}}$ term is of the 3rd degree in $n$, and is of the form

$$A + Bn + Cn(n+1) + Dn(n+1)(n+2) + E.2^{n-1}.$$

Giving to $n$ in succession the values 1, 2, 3, 4, 5 we obtain 5 equations to determine $A, B, C, D, E$. From these we find

$$A = 0, \quad B = -1, \quad C = 0, \quad D = \tfrac{2}{3}, \quad E = -6,$$

$\therefore$ the $n^{\text{th}}$ term is $\tfrac{2}{3}n(n+1)(n+2) - n - 6.2^{n-1}$,

$\therefore$ the sum of $n$ terms

$$= \tfrac{2}{3}.\tfrac{1}{4}.n(n+1)(n+2)(n+3) - \tfrac{1}{2}n(n+1) - 6(2^n - 1).$$

2. $a^3+b^3+c^3+3abc = (a+b+c)(a^2+b^2+c^2-bc-ca-ab)+6abc$

$$> 2c\left\{(a+b)^2 + c^2 - 3ab - c(a+b)\right\} + 6abc$$

$$> 2c\left\{(a+b)^2 + c^2 - c(a+b)\right\}$$

$$> 2c\left\{2c(a+b) - c(a+b)\right\} \text{ for } x^2+y^2 > 2xy$$

$$> 2c^2(a+b).$$

**3.** Let $ABC$ be the first triangle, $D$, $E$, $F$ the points of contact of the inscribed circle, $A'B'C'$ the second triangle.

Then $\quad a' = AE = s - a, \quad b' = s - b, \quad c' = s - c.$

Now $\quad \rho = \frac{S'}{s'}; \quad \rho' = \frac{a'b'c'}{4S'},$

$$\therefore 2\rho\rho' = \frac{a'b'c'}{2s'} = \frac{(s-a)(s-b)(s-c)}{3s-a-b-c} = \frac{s(s-a)(s-b)(s-c)}{s^2} = \frac{S^2}{s^2} = r^2.$$

**4.** $\quad \sin x = n\cos(x+a),$

$$\therefore \frac{1}{2i}\left\{\epsilon^{xi} - \epsilon^{-xi}\right\} = \frac{n}{2}\left\{\epsilon^{(x+a)i} + \epsilon^{-(x+a)i}\right\},$$

$$\therefore \epsilon^{xi}(1 - ni\epsilon^{ai}) = \epsilon^{-xi}(1 + ni\epsilon^{-ai}),$$

$$\therefore \epsilon^{2xi} = \frac{1 + ni\epsilon^{-ai}}{1 - ni\epsilon^{ai}},$$

$$\begin{aligned}
2xi &= \log(1 + ni\epsilon^{-ai}) - \log(1 - ni\epsilon^{ai}) \\
&= ni\epsilon^{-ai} - \tfrac{1}{2}.n^2.i^2.\epsilon^{-2ai} + \tfrac{1}{3}.n^3.i^3.\epsilon^{-3ai} - \ldots \\
&\quad + ni\epsilon^{+ai} + \tfrac{1}{2}n^2.i^2.\epsilon^{+2ai} + \tfrac{1}{3}.n^3.i^3.\epsilon^{+3ai} + \ldots \\
&= i\left\{n(\epsilon^{ai} + \epsilon^{-ai}) - \tfrac{1}{3}n^3(\epsilon^{3ai} + \epsilon^{-3ai}) + \ldots\right\} \\
&\quad - \left\{\tfrac{1}{2}n^2(\epsilon^{2ai} - \epsilon^{-2ai}) - \tfrac{1}{4}n^4(\epsilon^{4ai} - \epsilon^{-4ai}) + \ldots\right\} \\
&= 2i\left\{n\cos a - \tfrac{1}{3}n^3.\cos 3a + \ldots\right\} \\
&\quad - 2i\left\{\tfrac{1}{2}n^2\sin 2a - \tfrac{1}{4}n^4\sin 4a + \ldots\right\},
\end{aligned}$$

$$\therefore x = n\cos a - \tfrac{1}{3}n^3\cos 3a + \ldots - \left\{\tfrac{1}{2}n^2\sin 2a - \tfrac{1}{4}n^4\sin 4a + \ldots\right\}.$$

Let $\quad x = \frac{\pi}{8}, \quad a = \frac{\pi}{4}, \quad n = 1,$

$$\begin{aligned}
\therefore \frac{\pi}{8} &= \frac{1}{\sqrt{2}}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \ldots\right) - \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{10} - \frac{1}{14} + \ldots\right) \\
&= \frac{1}{\sqrt{2}}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \ldots\right) - \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) \\
&\quad \frac{1}{\sqrt{2}}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \ldots\right) - \frac{1}{2}\cdot\frac{\pi}{4},
\end{aligned}$$

$$\therefore \frac{\pi}{4} = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \dots\right),$$

$$\therefore \frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \dots$$

5. Let the tangents at $B$ and $C$ meet in $D$. Join $DA$, and produce it to meet $BC$ in $E$. Then since $D$ is the pole of $BC$, and $A$ the pole of $PT$, $\therefore$ $P$ is the pole of $AD$. $\therefore$ $BPCE$ is a harmonic range, and $\therefore$ also $TPQA$ is a harmonic range.

6. Let $E$ and $K$ be the middle points of $AB$ and $BC$, and $F$ the position of the peg on which $AB$ rests. Let $R$ denote the reaction at $B$. Let the direction of $R$ meet the verticals through $E$ and $K$ in $G$ and $H$.

Then from the equilibrium of $AB$,

$$R : W :: \sin EGF : \sin BGF :: \sin\alpha : \cos(\alpha + \beta).$$

From the equilibrium of $BC$,

$$R : W :: \sin KHC : \sin BHC :: \cos\beta : \sin 2\beta :: 1 : 2\sin\beta,$$

$$\therefore \sin\alpha : \cos(\alpha + \beta) :: 1 : 2\sin\beta,$$

$$\therefore 2\sin\alpha\sin\beta = \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

$$\therefore 3\tan\alpha\tan\beta = 1.$$

7. The given equation can be written

$$\left\{\frac{2x}{a}\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{y}{b}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\right\}^2$$

$$+ 2\left\{\frac{x}{a}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) + \frac{2y}{b}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)$$

$$- 2\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)^2 = 0.$$

Divide by $\cos^4\frac{\theta}{2}$, and for $\tan\frac{\theta}{2}$ write $z$,

$$\therefore \left\{2\frac{x}{a}z - \frac{y}{b}(1 - z^2)\right\}^2 + 2\frac{x}{a}(1 - z^4) + 4\frac{y}{b}(z + z^3) - 2(1 + z^2)^2 = 0,$$

$$\therefore z^4\left(\frac{y^2}{b^2} - \frac{2x}{a} - 2\right) + \frac{4y}{b}\left(\frac{x}{a} + 1\right)z^3 + 2\left(\frac{2x^2}{a^2} - \frac{y^2}{b^2} - 2\right)z^2$$

$$+ \frac{4y}{b}\left(1 - \frac{x}{a}\right)z + \frac{y^2}{b^2} + \frac{2x}{a} - 2 = 0.$$

To find the envelope we must express the condition that the above expression may be a perfect square.

Now $(z^2 + dz + e)^2 = z^4 + 2dz + (2e + d^2)\, z^2 + 2edz + e^2$.

$\therefore$ absolute term = sq. of coef. of $z \div$ sq. of coef. of $z^3$,

$$\therefore \frac{\frac{y^2}{b^2} + \frac{2x}{a} - 2}{\frac{y^2}{b^2} - \frac{2x}{a} - 2} = \left\{ \frac{\frac{4y}{b}\left(1 - \frac{x}{a}\right)}{\frac{4y}{b}\left(1 + \frac{x}{a}\right)} \right\}^2,$$

$$\therefore \left(1 + \frac{x}{a}\right)^2 \left(\frac{y^2}{b^2} + \frac{2x}{a} - 2\right) = \left(1 - \frac{x}{a}\right)^2 \left(\frac{y^2}{b^2} - \frac{2x}{a} - 2\right),$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

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## Paper LXIII.

1. Let $30\sqrt{3} + 37 = (\sqrt{y} + x)^3$; $\therefore 30\sqrt{3} - 37 = (\sqrt{y} - x)^3$,

$\therefore$ multiplying, $1331 = (y - x^2)^3$, $\therefore 11 = y - x^2$,

and $30\sqrt{3} + 37 = x^3 + 3xy + (3x^2 + y)\sqrt{y}$,

$\therefore 37 = x^3 + 3xy = x^3 + 3x(x^2 + 11)$,

$\therefore 4x^3 + 33x - 37 = 0$. $x = 1$ evidently satisfies this equation.

$$\therefore (x - 1)(4x^2 + 4x + 37) = 0,$$

$$\therefore x = 1,\ y = 12,\ \therefore \sqrt[3]{30\sqrt{3} + 37} = 2\sqrt{3} + 1.$$

Let $a + \sqrt{b} + \sqrt{c} + \sqrt{d} = (x + \sqrt{y} + \sqrt{z})^3$

$= x^3 + 3x(y + z) + \sqrt{y}(3x^2 + y + 3z) + \sqrt{z}(3x^2 + z + 3y) + 6x\sqrt{yz}$.

Put $a = x^3 + 3x(y + z)$; $\sqrt{b} = \sqrt{y}(3x^2 + y + 3z)$;

$$\sqrt{c} = \sqrt{z}(3x^2 + z + 3y);\quad \sqrt{d} = 6x\sqrt{yz}.$$

On substituting, we see at once that $\sqrt{\frac{bc}{d}}$, $\sqrt{\frac{cd}{b}}$, $\sqrt{\frac{db}{c}}$ are rational.

Now $k = \sqrt{\frac{bc}{d}} = \frac{1}{6x}(3x^2 + y + 3z)(3x^2 + z + 3y)$,

$\therefore 6xk = 9x^4 + 12x^2(y + z) + 3(y + z)^2 + 4yz$

$= 9x^4 + 4x(a - x^3) + \frac{(a - x^3)^2}{3x^2} + \frac{d}{9x^2}$,

$\therefore 54x^3k = 45x^6 + 36ax^3 + 3(a^2 - 2ax^3 + x^6) + d$,

or $48t^2 + 6(5a - 9k)t + 3a^2 + d = 0$, where $t = x^3$ . . . . $(A)$

Now let $16 + 14\sqrt{2} + 12\sqrt{3} + 6\sqrt{6} = (x + \sqrt{y} + \sqrt{z})^3$

$= x^3 + 3x(y + z) + \sqrt{y}(3x^2 + y + 3z) + \sqrt{z}(3x^2 + 3y + z) + 6x\sqrt{yz}$,

$\therefore 16 = x^3 + 3x(y + z)$; $14\sqrt{2} = \sqrt{y}(3x^2 + y + 3z)$

$12\sqrt{3} = \sqrt{z}(3x^2 + 3y + z)$; $6\sqrt{6} = 6x\sqrt{yz}$,

$\therefore$ if $k = \sqrt{\frac{392.432}{216}} = \sqrt{784} = 28$,

the above equation $(A)$ becomes

$48t^2 + 6(80 - 252)t + 768 + 216 = 0$,

$\therefore 48t^2 - 1032t + 984 = 0$. Now $t = 1$ obviously satisfies this.

$\therefore x^3 = 1$; $\therefore x = 1$; and $15 = 3(y + z)$ $\therefore y + z = 5$; and $yz = 6$,

$\therefore y = 3$; $z = 2$,

$\therefore \sqrt[3]{16 + 14\sqrt{2} + 12\sqrt{3} + 6\sqrt{6}} = 1 + \sqrt{2} + \sqrt{3}$.

2. If $x'$, $y'$, $z'$ are small quantities, the expression

$$\frac{\cos(x + x') - \cos(y + y') - \cos(z + z')}{\sin(y + y')\sin(z + z')}$$

$$= \frac{\cos x - x'\sin x - \frac{1}{2}\{\cos(y + z + y' + z') + \cos(y - z + y' - z')\}}{\frac{1}{2}\{\cos(y - z + y' - z') - \cos(y + z + y' + z')\}}$$

$$= \frac{\cos x - x'\sin x - \frac{1}{2}\{\cos(y + z) - (y' + z')\sin(y + z)}{\frac{1}{2}\{\cos(y - z) - (y' - z')\sin(y - z) - \cos(y + z)} \quad \frac{+ \cos(y - z) - (y' - z')\sin(y - z)\}}{+ (y' + z')\sin(y + z)\}}$$

$$= \frac{\cos x - x' \sin x - \cos y \cos z + y' \sin y \cos z + z' \cos y \sin z}{\sin y \sin z + y' \cos y \sin z + z' \sin y \cos z} \quad \ldots (A)$$

Now if this expression $= \frac{\cos x - \cos y \cos z}{\sin y \sin z}$ each of them will also

$$= \frac{- x' \sin x + y' \sin y \cos z + z' \cos y \sin z}{y' \cos y \sin z + z' \sin y \cos z} \quad \ldots\ldots (B)$$

Now if $\frac{x'}{\tan \frac{x}{2} (\cos y + \cos z)} = \frac{y'}{\sin y} = \frac{z'}{\sin z}$, the expression $(B)$

$$= \frac{- \sin x \tan \frac{x}{2} (\cos y + \cos z) + \sin^2 y \cos z + \cos y \sin^2 z}{\sin y \cos y \sin z + \sin y \sin z \cos z}$$

$$= \frac{- (1 - \cos x) (\cos y + \cos z) + (1 - \cos^2 y) \cos z + (1 - \cos^2 z) \cos y}{\sin y \sin z (\cos y + \cos z)}$$

$$= \frac{(\cos x - \cos y \cos z) (\cos y + \cos z)}{\sin y \sin z (\cos y + \cos z)} = \frac{\cos x - \cos y \cos z}{\sin y \sin z},$$

$\therefore$ if $x'y'z'$ are in the given proportion, $\frac{\cos x - \cos y \cos z}{\sin y \sin z} = B$, and by adding numerator and denominator, each is evidently $= A$.

3. $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \log_e (1 + 1) = \log_e 2 = \cdot 6931471 \ldots$

4. The points $A$, $B$ are supposed to be fixed. Draw the tangent $BH$. Then the angle $BFG = ECD = EAB$, and $\therefore$ the triangles $FBG$, $EBA$ are similar. $\therefore$ $BG : BF :: BE : BA$. $\therefore$ $BG \cdot BA = BH^2$. $\therefore$ $G$ is a fixed point.

5. Let $P$ be the point on the ellipse such that $FP$ is a minimum. Take a point $P'$ on the curve indefinitely near to $P$. Then $FP = FP'$. $\therefore$ the angle $FPP' = FP'P$, and each ultimately $= 1$ right angle. But $PP'$ is the direction of the tangent at $P$. $\therefore$ $FP$ is the normal.

$$\therefore SF : SP :: SA : AX, \ \therefore SP = \frac{1}{e} \cdot SF.$$

6. Suppose the weight of the triangle to be represented by $a + b + c$. Then $\frac{1}{3}$ of this is supported at each angular point, and at these points we have remaining the pressures represented by $a$, $b$, $c$. Divide $AB$

at $D$ so that $AD : DB :: b : a :: AC : CB$. Then the point where the weight is placed must be in $CD$, which bisects the angle $C$. $\therefore$ the required point is the centre of the inscribed circle.

7. Let $PSP'$ be the focal chord, $QQ'$ the parallel diameter. Then $QQ'^2 = PP' . AA'$. Let $p$, $p'$, $q$, $q'$ be points on the auxiliary circle corresponding to $P$, $P'$, $Q$, $Q'$. Then $qq'$ is evidently a diameter of the circle. $\therefore pp' : qq' :: PP' : QQ' :: QQ' : AA'$. But $qq' = AA'$. $\therefore QQ' = pp'$.

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## PAPER LXIV.

1. Let $a^2 - \frac{1}{x} = b^2 - \frac{1}{y} = c^2 - \frac{1}{z} = k$, $\therefore a^2 = k + \frac{1}{x}$, &c.

and (1) $a^2x^2 + b^2y^2 + c^2z^2 = 0$,

(2) $a^2x^3 + b^2y^3 + c^2z^3 = 0$,

$\therefore$ (3) $a^4x^3 + b^4y^3 + c^4z^3 = a^2x^3 . a^2 + b^2y^3 . b^2 + c^2z^3 . c^2$

$$= a^2x^3\left(k + \frac{1}{x}\right) + b^2y^3\left(k + \frac{1}{y}\right) + c^2z^3\left(k + \frac{1}{z}\right)$$

$$= k\,(a^2x^3 + b^2y^3 + c^2z^3) + a^2x^2 + b^2y^2 + c^2z^2$$

$$= 0.$$

$\therefore$ from (2) and (3) $\frac{a^2x^3}{b^2 - c^2} = \frac{b^2y^3}{c^2 - a^2} = \frac{c^2z^3}{a^2 - b^2}$,

$$\therefore \frac{a^3x^3}{a\,(b^2 - c^2)} = \frac{b^3y^3}{b\,(c^2 - a^2)} = \frac{c^3z^3}{c\,(a^2 - b^2)},$$

$\therefore$ substituting in (1) we have

$$a^{\frac{3}{2}}(b^2 - c^2)^{\frac{3}{2}} + b^{\frac{3}{2}}(c^2 - a^2)^{\frac{3}{2}} + c^{\frac{3}{2}}(a^2 - b^2)^{\frac{3}{2}} = 0.$$

2. The required sum $= \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots$

$$+ \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \ldots$$

$$+ \ldots\ldots\ldots\ldots$$

$$= \frac{\frac{1}{2^2}}{1 - \frac{1}{2}} + \frac{\frac{1}{3^2}}{1 - \frac{1}{3}} + \dots$$

$$= \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots$$

$$= 1.$$

3. If $a = 50, \quad b = 41, \quad c = 21,$

then $s = 56, \quad s - a = 6, \quad s - b = 15, \quad s - c = 35.$

Let $x$ be the length to be added to $b$, $y$ the length to be added to $c$.

Then $x + y = -1.$

Now $S^2 = s(s-a)(s-b)(s-c) = s(s-a')(s-b')(s-c').$

$\therefore 56.6.15.35 = 56.5(15 - x)(35 - y),$

$\therefore xy - 35x - 15y = 105;$ and $y = -(x+1),$

$\therefore x^2 + 21x + 90 = 0, \quad \therefore x = -15 \text{ or } -6, \quad y = 14 \text{ or } 5.$

$\therefore$ the sides are either 51, 26, 35; or 51, 35, 26.

4. $$\frac{b^{\frac{3}{2}} + c^{\frac{3}{2}}}{b^{\frac{1}{2}} c^{\frac{1}{2}}} \cos A = \frac{1}{a^{\frac{1}{2}} b^{\frac{1}{2}} c^{\frac{1}{2}}} (a^{\frac{1}{2}} b^{\frac{3}{2}} + a^{\frac{1}{2}} c^{\frac{3}{2}}) \cdot \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{1}{2a^{\frac{3}{2}} b^{\frac{3}{2}} c^{\frac{3}{2}}} a^{\frac{1}{2}} (b^{\frac{3}{2}} + c^{\frac{3}{2}})(b^2 + c^2 - a^2)$$

$$= \frac{1}{2a^{\frac{3}{2}} b^{\frac{3}{2}} c^{\frac{3}{2}}} (a^{\frac{3}{2}} b^{\frac{7}{2}} + a^{\frac{3}{2}} b^{\frac{3}{2}} c^2 - a^{\frac{1}{2}} b^{\frac{3}{2}} + a^{\frac{3}{2}} b^2 c^{\frac{3}{2}} + a^{\frac{3}{2}} c^{\frac{7}{2}} - a^{\frac{7}{2}} c^{\frac{3}{2}}).$$

By symmetric changes of the letters we can write down the other two terms of the given expression. $\therefore$ by addition we find it

$$= \frac{1}{2a^{\frac{3}{2}} b^{\frac{3}{2}} c^{\frac{3}{2}}} (2a^2 b^{\frac{3}{2}} c^{\frac{3}{2}} + 2a^{\frac{3}{2}} b^2 c^{\frac{3}{2}} + 2a^{\frac{3}{2}} b^{\frac{3}{2}} c^2)$$

$$= a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}}.$$

5. By the *Theory of Maximum and Minimum* if we suppose $MN$ to remain fixed we know that $NL$ and $LM$ make equal angles with $BC$, and similarly for each of the other pairs of sides of the triangle $LMN$. $\therefore$ when $LMN$ has a min. perimeter, any two sides make equal angles with the corresponding side of $ABC$. Now we know that this is the case with the triangle whose vertices are the feet of the perpendiculars. $\therefore$ this triangle gives the required position for $LMN$.

6. Let $AB$ represent the boy, $CB$ the chair, and let $ABC = \theta$, $AB = l$.

Let $R$, $R'$ denote the reactions at $A$ and $B$.

Then resolving vertically $R = W$.

Taking moments about $A$, $R' . l\cos\theta = W . \frac{l}{2} . \sin\theta$, $\therefore R' = \frac{W}{2}\tan\theta$.

The boy will begin to slide when the friction at $A$ is a max. and $= \mu R = \mu W$.

Resolving horizontally, we have the friction $= R'$. $= \frac{W}{2}\tan\theta$,

$$\therefore \mu W = \frac{W}{2}\tan\theta, \quad \therefore \tan\theta = 2\mu,$$

$\therefore$ the boy will begin to slide when $\theta = \tan^{-1} 2\mu$.

The chair will begin to slide when $R' = \mu W'$, or $\tan\theta = 2\mu\frac{W'}{W}$.

The greatest possible value of $\theta$ which is consistent with equilibrium is the smaller of these two.

7. Let the asymptotes meet the directrix in $D$ and $D'$, and draw $SF$ parallel to $CD'$ meeting the directrix in $F$. Let $SR$ be the semi-latus rectum. Then from the similar triangles $SFX$, $CDX$,

$$SF : SX :: CD : CX :: CA : CX :: SA : AX :: SR : SX.$$

$\therefore SF = SR$. Let $SF$ meet the curve in $P$, and draw the ordinate $PN$, and $PK$ perpendicular to the directrix.

Then $SP : PK :: SA : AX :: SR : SX :: SF : SX :: SP : SN$,

$\therefore SN = PK = NX$. $\therefore P$ is the middle point of $SF$.

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## Paper LXV.

1. (1) $S = -1 - 3 + 3 + 23 + 63 + 129 + \ldots$

$S_1 = \quad -2 + 6 + 20 + 40 + 66 + \ldots$

$S_2 = \quad 8 + 14 + 20 + 26 + \ldots$

$S_3 = \quad 6 + 6 + 6 + \ldots$

Here the 3rd difference series gives us a series of equal terms.

$\therefore$ the $n^{\text{th}}$ term is of the 3rd degree in $n$, and is of the form

$$A + Bn + Cn(n+1) + Dn(n+1)(n+2).$$

$\therefore$ putting $n$ in succession $= 1, 2, 3, 4$ we have four equations to determine $A, B, C, D$, from which we find

$$A = +3, \quad B = 0, \quad C = -5, \quad D = 1,$$

$\therefore$ the $n^{\text{th}}$ term is

$$n(n+1)(n+2) - 5n(n+1) + 3.$$

$\therefore$ the sum of $n$ terms

$$= \tfrac{1}{4}n(n+1)(n+2)(n+3) - \tfrac{5}{3}n(n+1)(n+2) + 3n.$$

(2) The $n^{\text{th}}$ term of the series $= \dfrac{7n-3}{(5n-2)(5n+3)(5n+8)}$

$$= \frac{\frac{7}{5}(5n-2) - \frac{1}{5}}{(5n-2)(5n+3)(5n+8)} = \frac{\frac{7}{5}}{(5n+3)(5n+8)} - \frac{\frac{1}{5}}{(5n-2)(5n+3)(5n+8)},$$

$\therefore$ the sum of $n$ terms, [see LXII., 1, (2)]

$$= C - \frac{7}{5}\cdot\frac{1}{5}\cdot\frac{1}{5n+8} + \frac{1}{5}\cdot\frac{1}{5.2}\cdot\frac{1}{(5n+3)(5n+8)},$$

Now when $n = 0$, the sum $= 0$,

$$\therefore C = \frac{7}{25.8} - \frac{1}{50.3.8} = \frac{41}{1200},$$

$\therefore$ the sum of $n$ terms

$$= \frac{1}{50}\left\{\frac{41}{24} - \frac{14}{5n+8} + \frac{1}{(5n+3)(5n+8)}\right\}.$$

(3) The given series $= S_1 + S_2$

where $S_1 = \frac{1 \cdot 2^3}{\lfloor 3} + \frac{2 \cdot 2^4}{\lfloor 4} + \frac{3 \cdot 2^5}{\lfloor 5} + \ldots + \frac{n \cdot 2^{n+2}}{\lfloor n+2}$

and $S_2 = \frac{1}{\lfloor 2} + \frac{2}{\lfloor 3} + \frac{3}{\lfloor 4} + \ldots + \frac{n}{\lfloor n+1}$.

Now $S_1 = \frac{(3-2)\,2^3}{\lfloor 3} + \frac{(4-2)\,2^4}{\lfloor 4} + \ldots + \frac{(n+2-2) \cdot 2^{n+2}}{\lfloor n+2}$

$$= \frac{2^3}{\lfloor 2} + \frac{2^4}{\lfloor 3} + \frac{2^5}{\lfloor 4} + \ldots + \frac{2^{n+2}}{\lfloor n+1}$$
$$- \frac{2^4}{\lfloor 3} - \frac{2^5}{\lfloor 4} - \ldots - \frac{2^{n+2}}{\lfloor n+1} - \frac{2^{n+3}}{\lfloor n+2}$$

$$= \frac{2^3}{\lfloor 2} - \frac{2^{n+3}}{\lfloor n+2},$$

$$S_2 = \frac{2-1}{\lfloor 2} + \frac{3-1}{\lfloor 3} + \frac{4-1}{\lfloor 4} + \ldots + \frac{n+1-1}{\lfloor n+1}$$

$$= \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots + \frac{1}{\lfloor n}$$
$$- \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} - \ldots - \frac{1}{\lfloor n} - \frac{1}{\lfloor n+1}$$

$$= 1 - \frac{1}{\lfloor n+1}.$$

$\therefore$ the sum of $n$ terms of the given series

$$= 5 - \frac{2^{n+3}}{\lfloor n+2} - \frac{1}{\lfloor n+1}.$$

2. $$(1 - \epsilon^x)^n = \left\{x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\right\}^n (-1)^n \quad \ldots \quad (A)$$

$$= 1 - n\epsilon^x + \frac{n(n-1)}{\lfloor 2}\,\epsilon^{2x} \quad \ldots\ldots \quad (B)$$

Equate the right-hand members of $(A)$ and $(B)$, and multiply both sides by $\epsilon^{nx}$.

$$\therefore \epsilon^x - n\epsilon^{2x} + \frac{n(n-1)}{\lfloor 2} \epsilon^{3x} \ldots$$

$$= (-1)^n \left\{x + \frac{x^2}{\lfloor 2} + \ldots\right\}^n (1 + x + \frac{x^2}{\lfloor 2} + \ldots),$$

$\therefore$ equating coefficients of $x^n$ we have

$$\frac{1}{\lfloor n} - n \cdot \frac{2^n}{\lfloor n} + \frac{n(n-1)}{\lfloor 2} \cdot \frac{3^n}{\lfloor n} - \ldots = (-1)^n,$$

$$\therefore 1^n - n \cdot 2^n + \frac{n(n-1)}{\lfloor 2} \cdot 3^n - \ldots = (-1)^n \lfloor n.$$

3. Let $C = \sin^n \phi \cos n\theta + n \sin^{n-1} \phi \cos(n-1)\theta \sin(\theta - \phi) + \ldots$

$S = \sin^n \phi \sin n\theta + n \sin^{n-1} \phi \sin(n-1)\theta \sin(\theta - \phi) + \ldots$

$\therefore C + Si = \sin^n \phi\, \epsilon^{n\theta i} + n \sin^{n-1} \phi\, \epsilon^{(n-1)\theta i} \cdot \sin(\theta - \phi) + \ldots$

$= \{\sin\phi \cdot \epsilon^{\theta i} + \sin(\theta - \phi)\}^n$

$= \{\sin\phi(\cos\theta + i\sin\theta) + \sin\theta\cos\phi - \cos\theta\sin\phi\}^n$

$= \{\sin\theta(\cos\phi + i\sin\phi)\}^n$

$= \sin^n\theta\, \epsilon^{n\phi i} = \sin^n\theta(\cos n\phi + i\sin n\phi).$

$\therefore C = \sin^n\theta \cos n\phi.$

4. Let $DE$ cut the circle in $F$.

Then $DB \cdot DC^2 = AD \cdot AB^2 = AD \cdot DE^2$

and $DB \cdot DC = DE \cdot DF,$

$\therefore DC : DA :: DE : DF.$ $\therefore$ $AF$ and $CE$ are parallel.

$\therefore$ the angle $FAB = ECB = \frac{1}{2} EAB = EAF.$

And $AEF = AFE = FAD + FDA$

$= FAD + EAD$, since $DE = EA.$

Now $AEF = \pi - (EAD + EDA) = \pi - 2EAD$

$\therefore \pi - 2EAD = \frac{1}{2} EAD + EAD = \frac{3}{2} EAD,$

$\therefore EAB = \frac{1}{7} \cdot 2\pi.$ $\therefore$ arc $EB = \frac{1}{7}$ of circumference.

5. This question will be found treated fully (with the exception of the last point), in Smith's *Conics*, pp. 234, 5, 6. It only remains to prove that the rectangular hyperbola will pass through the centre of the circle. With the notation given in Smith, the line through

$$\left\{\frac{1}{2}\left(\frac{1}{a}+\frac{1}{a'}\right),\ 0\right\}$$ at right angles to the axis of $x$ is

$$x + y\cos\theta = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{a'}\right) \quad \ldots\ldots\ldots \quad (1)$$

and the line through $\left\{0,\ \frac{1}{2}\left(\frac{1}{b}+\frac{1}{b'}\right)\right\}$ at right angles to the axis of $y$ is

$$y + x\cos\theta = \frac{1}{2}\left(\frac{1}{b}+\frac{1}{b'}\right) \quad \ldots\ldots\ldots \quad (2)$$

and these two lines evidently pass through the centre of the circle. Multiply (1) by $x$, and (2) by $y$, and subtract, and we have

$$x^2 - y^2 - \frac{1}{2}\left(\frac{1}{a}+\frac{1}{a'}\right)x + \frac{1}{2}\left(\frac{1}{b}+\frac{1}{b'}\right)y = 0$$

which is the equation to the rectangular hyperbola passing through the 9 points, and which $\therefore$ also passes through the centre of the circle.

6. Let one bullet be fired from $A$ in direction of another which is being let fall from $B$, and let the first strike the vertical through $B$ after a time $t$. It will then be at a distance $\frac{1}{2}gt^2$ below $B$. But this is the distance through which the second bullet has fallen in the same time. $\therefore$ the two will meet.

If they coalesce, the horizontal momentum is unaltered, and since the mass is doubled, the horizontal velocity is reduced to one-half its former value.

Now (horizontal velocity)$^2 = 2g \cdot \frac{\text{lat. rect.}}{4}$.

$\therefore$ the latus rectum of the joint path is $\frac{1}{4}$ that of the former.

7. $$a' = a\cos^2\theta + b\sin^2\theta + 2h\sin\theta\cos\theta,$$

$$b' = a\sin^2\theta + b\cos^2\theta - 2h\sin\theta\cos\theta,$$

$$\therefore a' - b' = (a-b)\cos 2\theta + 2h\sin 2\theta$$

$$= 2h\sin 2\theta\left\{1 + \frac{a-b}{2h}\cot 2\theta\right\}$$

$$= 2h\sin 2\theta\left\{1 + \cot^2 2\theta\right\} = 2h\operatorname{cosec} 2\theta,$$

and since $2\theta$ is $< \pi$, cosec $2\theta$ is essentially positive, $\therefore$ the sign of $a' - b'$ is the same as the sign of $h$.

$$13x^2 + 2xy + 13y^2 - 22x + 50y - 23 = 0.$$

The coordinates of the centre are given by

$$26x + 2y = 22\,; \quad 2x + 26y = -50. \quad \therefore x = 1,\ y = -2.$$

$\therefore$ transferring to $(1, -2)$ the equation becomes

$$13x^2 + 2xy + 13y^2 = 84.$$

To determine the lengths of the axes we have

$$\alpha + \beta = 26,\ \alpha\beta = 169 - 1 = 168. \quad \therefore \alpha = 14,\ \beta = 12,$$

$$\therefore 14x^2 + 12y^2 = 84,\ \therefore \frac{x^2}{6} + \frac{y^2}{7} = 1,$$

which is the equation to an ellipse of which the semi-axes are $\sqrt{6}$, $\sqrt{7}$.

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## Paper LXVI.

1. The equations may be written

$$l(x - z) + ay + al\,xy = 0 \quad \ldots\ldots\ldots \quad (1)$$

$$m(x - z) + by + bm\,xy = 0 \quad \ldots\ldots\ldots \quad (2)$$

$$n(x - z) + cy + cn\,xy = 0 \quad \ldots\ldots\ldots \quad (3)$$

From (2) and (3) we obtain

$$\frac{x - z}{bc(m - n)} = \frac{y}{mn(c - b)} = \frac{xy}{bn - cn} \quad \ldots\ldots \quad (4)$$

If (1) is also true, we have by substitution in (1) from (4)

$$0 = bcl(m - n) + amn(c - b) + al(bn - cm)$$

$$= bc\left(\frac{1}{n} - \frac{1}{m}\right) + \frac{a}{l}(c - b) + a\left(\frac{b}{m} - \frac{c}{n}\right)$$

$$= (b - c)\frac{a}{l} + (c - a)\frac{b}{m} + (a - b)\frac{c}{n}.$$

From (4)

$$\frac{y}{mn(c - b)} = \frac{xy}{bn - cm},$$

$\therefore$ either $y = 0$, in which case also $x - z = 0$ [from (4)]

or $$x = \frac{cm - bn}{mn(b - c)} = \frac{\frac{c}{n} - \frac{b}{m}}{b - c},$$

and from the equation $(1 + lx)(1 + ay) = 1 + lz$

we see that $y = -\frac{1}{a}$, $z = -\frac{1}{l}$ are solutions, and two other sets can be obtained from the two other equations, viz.

$$y = -\frac{1}{b},\ z = -\frac{1}{m};\quad y = -\frac{1}{c},\ z = -\frac{1}{n}.$$

2. $$r^2 = \frac{a^2b^2\cos^2(\theta - \phi)}{a^2\sin^2\phi + b^2\cos^2\phi} = \frac{a^2b^2(\cos\theta\cos\phi + \sin\theta\sin\phi)^2}{a^2\sin^2\phi + b^2\cos^2\phi}$$

$$= \frac{a^2b^2(\cos\theta + \sin\theta\tan\phi)^2}{a^2\tan^2\phi + b^2} = \frac{a^2b^2(\cos\theta - \frac{b^2}{a^2}\sin\theta\cot\phi')^2}{a^2 \cdot \frac{b^4}{a^4}\cot^2\phi' + b^2}$$

$$= \frac{(a^2\cos\theta\sin\phi' - b^2\sin\theta\cos\phi')^2}{a^2\sin^2\phi' + b^2\cos^2\phi'}.$$

Also $$r^2 = \frac{a^2b^2\cos^2(\theta - \phi')}{a^2\sin^2\phi' + b^2\cos^2\phi'}$$

$$\therefore 2r^2 = \frac{a^2b^2(\cos\theta\cos\phi' + \sin\theta\sin\phi')^2 + (a^2\cos\theta\sin\phi' - b^2\sin\theta\cos\phi')^2}{a^2\sin^2\phi' + b^2\cos^2\phi'}$$

$$= \frac{(a^2\cos^2\theta + b^2\sin^2\theta)(a^2\sin^2\phi' + b^2\cos^2\phi')}{a^2\sin^2\phi' + b^2\cos^2\phi'}$$

$$= a^2\cos^2\theta + b^2\sin^2\theta.$$

3. Write the first equation

$$a^3\sin\phi(y + b\sin\phi) = -b^3\cos\phi(x + a\cos\phi) \quad . \quad . \quad (A)$$

and $$ax\sec\phi - by\operatorname{cosec}\phi = a^2 - b^2,$$

$$\therefore a\left(\frac{x}{\cos\phi} - a\right) = b\left(\frac{y}{\sin\phi} - b\right),$$

$$\therefore a\sin\phi(x - a\cos\phi) = b\cos\phi(y - b\sin\phi) \quad . \quad . \quad (B)$$

From $(A)$ and $(B)$ we have

$$a^3b \sin\phi \cos\phi\,(y^2 - b^2\sin^2\phi) = -ab^3 \sin\phi\cos\phi\,(x^2 - a^2\cos^2\phi,$$

$$\therefore a^2y^2 - a^2b^2\sin^2\phi = -b^2x^2 + a^2b^2\cos^2\phi,$$

$$\therefore a^2y^2 + b^2x^2 = a^2b^2(\sin^2\phi + \cos^2\phi) = a^2b^2.$$

4. Let $BG$ meet $AC$ in $L$, and let $CE$ meet $AB$ in $K$, and let the perpendicular from $A$ meet $BC$ in $H$. Then from the similar triangles $AKC$, $BKE$, we have $AC : AK :: BE : BK$, and $BE = AB$,

$$\therefore AB : AC :: BK : AK.$$

From $ABL$, $CGL$, $AB : AC :: AL : LC$.

From $ACH$, $ABC$, $AC : BC :: CH : AC$.

From $ABH$, $ABC$, $BC : AB :: AB : BH$.

$\therefore$ compounding these four ratios we have

$$AB^2 . AC^2 . BC . AK . BH . CL = AB^2 . AC^2 . BC . AL . CH . BK,$$

$$\therefore AK . BH . CL = AL . CH . BK,$$

$\therefore$ the three lines $AH$, $BL$, $CK$ are concurrent.

5. Let the tangent parallel to $PQ$ meet $TP$ in $p$ and $TQ$ in $q$.

Then $p$, $q$ are the middle points of $TP$ and $TQ$. $\therefore$ $Op$, $Oq$ are at right angles to $TP$ and $TQ$. $\therefore$ $TO$ is the diameter of the circle which passes through $T$, $p$, $q$. But this circle also passes through $S$. $\therefore$ $TSO$ is a right angle.

6. Let $Op$, $Oq$, $Or$ represent the forces. Let the transversal cut $rp$ produced in $K$.

Then $$\frac{OL}{Lp} = \frac{OM}{pK}, \quad \therefore \frac{Op}{OL} = \frac{pK + OM}{OM} = \frac{Kr - pr + OM}{OM},$$

$$\therefore \frac{Op}{OL} + \frac{Oq}{OM} = 1 + \frac{Kr}{OM} = 1 + \frac{Nr}{ON} = \frac{Or}{ON},$$

$$\therefore \frac{P}{OL} + \frac{Q}{OM} = \frac{R}{ON}.$$

7. Let the ball $A$ moving with velocity $V$ strike the ball $B$, and let its directions before and after collision make angles $\phi$, $\alpha$ respectively with the line of centres. After collision let $u$, $v$ be the components of the velocity of $A$ parallel and perpendicular to the line of centres, and let $u'$ be the velocity of $B$ after being struck.

Then $V\cos\phi = u + u'$; $eV\cos\phi + u = u'$. $\therefore 2u = (1 - e)V\cos\phi$, and $v = V\sin\phi$, for the momentum of $A$ perpendicular to the line of centres is unaltered by the collision.

$$\therefore \frac{2}{1-e}\tan\phi = \frac{v}{u} = \tan a,$$

$$\therefore \tan\phi = \tfrac{1}{2}(1 - e)\tan a.$$

Now when the deviation is a max. the angle between the old and new directions $= \frac{\pi}{2}$,

$$\therefore \phi - \frac{\pi}{2} = \pi - a. \quad \therefore a = 2\pi - \left(\frac{\pi}{2} + \phi\right).$$

$$\therefore \tan a = -\tan\left(\frac{\pi}{2} + \phi\right) = \cot\phi,$$

$$\therefore \tan^2\phi = \tfrac{1}{2}(1 - e),$$

$$\therefore \sin^2\phi = \frac{1-e}{3-e} = \frac{c^2\sin^2\theta}{a^2},$$

$$\therefore \sin\theta = \frac{a}{c}\sqrt{\frac{1-e}{3-e}}.$$

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## Paper LXVII.

1. Let $x + c$ be the common measure.

Then $x^2 + ax + b = (x + c)\left(x + \frac{b}{c}\right) = x^2 + x\left(c + \frac{b}{c}\right) + b$

$$x^2 + a'x + b' = (x + c)\left(x + \frac{b'}{c}\right) = x^2 + x\left(c + \frac{b'}{c}\right) + b'.$$

$$\therefore c + \frac{b}{c} = a; \quad c + \frac{b'}{c} = a'; \quad \therefore \frac{b - b'}{c} = a - a'; \quad \therefore \frac{1}{c} = \frac{a - a'}{b - b'},$$

and $$c + \frac{b + b'}{c} = a + \frac{b'}{c} = a + \frac{b'(a - a')}{b - b'} = \frac{ab - a'b'}{b' - b'},$$

and $$c^2 = ac - b = \frac{a(b - b')}{a - a'} - b = \frac{a'b - ab'}{a - a'}.$$

$$\therefore\ b + b' + \frac{bb'}{c^2} = b + b' + \frac{bb'(a-a')}{a'b - ab'} = \frac{a'b^2 - ab'^2}{a'b - ab'}.$$

$$\therefore\ \text{L.C.M.} = (x+c)\left(x+\frac{b}{c}\right)\left(x+\frac{b'}{c}\right)$$

$$= x^3 + x^2\left(c + \frac{b+b'}{c}\right) + x\left(b + b' + \frac{bb'}{c^2}\right) + \frac{bb'}{c}$$

$$= x^3 + x^2 \cdot \frac{ab - a'b'}{b - b'} + x \cdot \frac{a'b^2 - ab'^2}{a'b - ab'} + \frac{bb'(a-a')}{b-b'}.$$

2. Let $A$ start at the rate of $a$ miles, and $B$ at the rate of $b$ miles an hour, and suppose that each increases his speed by $c$ miles an hour. Then at the end of 1st hour $A$ has gone $a + \frac{3c}{2}$ miles, and $B$ has gone $b + \frac{c}{2}$ miles. $\therefore$ the distance between them is $b - a - c = \frac{3}{16}$ . . (1)

At the end of 2nd hour $A$ has gone $a + \frac{3c}{2} + a + \frac{11c}{2}$, or $2a + 7c$.

" $B$ " $b + \frac{c}{2} + b + \frac{5c}{2}$, or $2b + 3c$.

$\therefore$ the distance between them is $2b - 2a - 4c = \frac{1}{4}$ . . . . . (2)

From (1) and (2) $c = \frac{1}{16}$, $b - a = \frac{1}{4}$.

$\therefore$ $B$ starts at the rate of $\frac{1}{4}$ mile an hour quicker than $A$. $\therefore$ if $a$ be supposed given, since $A$ goes at an average rate of $a + \frac{15}{32}$ miles an hour, the distance from $P$ to $Q$ is $4a + \frac{15}{8}$ miles.

3. By Todh. *Trig.* cap. xxiv. Ex. 17

$$S^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\frac{\theta}{2}$$

where $\theta$ is the sum of two opposite angles. $\therefore$ $S$ is a max. when $\cos\frac{\theta}{2}$ is a min., *i.e.* when $\cos\frac{\theta}{2} = 0$. $\therefore$ $\frac{\theta}{2} = \frac{\pi}{2}$, $\therefore$ $\theta = \pi$, $\therefore$ the max. quadrilateral with given sides is that which can be inscribed in a circle, and its area is

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Again, since a circle can be inscribed in the quadrilateral,

$$\therefore\ a + c = b + d,\ \therefore\ 2s = a + b + c + d = 2(a+c) \text{ or } = 2(b+d),$$

$$\therefore\ s-a=c,\ s-b=d,\ s-c=a,\ s-d=b.$$

$$\therefore\ S=\sqrt{(s-a)(s-b)(s-c)(s-d)}=\sqrt{abcd}.$$

Now if $r$ be radius of inscribed circle, $r(a+b+c+d)=2\,.$ area

$$\therefore\ r=\frac{2\sqrt{abcd}}{a+b+c+d}=\frac{\sqrt{abcd}}{a+c},\ \text{or}=\frac{\sqrt{abcd}}{b+d},$$

$$\therefore\ r^2=\frac{abcd}{(a+c)(b+d)}.$$

The following is a geometrical proof of the theorem that the maximum quadrilateral which can be formed from four given straight lines is that about which a circle can be described.

Let $ABCD$ be the position of max. area. Take $ABC'D'$ a consecutive position keeping $AB$ fixed. Let $AD$, $BC$ meet in $O$.

Then since $AD=AD'$, and the angle $DAD'$ is small,

$\therefore$ the angle $ADD'$ is ultimately a right angle.

$\therefore$ the angle $ODD'$ is a right angle. And since the angle $DOD'$ is small,

$\therefore$ the angle $OD'D$ is a right angle.

$\therefore\ OD=OD'$; similarly $OC=OC'$. And the base $CD=$ base $C'D'$,

$\therefore$ the angle $DOC=D'OC'$, and $\therefore$ the triangle $OCD=OC'D'$.

$\therefore$ the angle $DOD'=COC'$, and the triangles $DOD'$, $COC'$ are similar.

$\therefore\ CC' : DD' :: OC : OD.$

Now since the triangle $OCD=OC'D'$, and the area $ABCD=ABC'D'$,

$\therefore$ the triangle $OAB=$ area $OD'ABC'$.

Take away the common part $OABC'$,

$\therefore$ the remainder $OAD'=OBC'$,

$\therefore\ OA\,.\,DD'=OB\,.\,CC'$,

$$\therefore\ \frac{OA}{OB}=\frac{CC'}{DD'}=\frac{OC}{OD},$$

$\therefore\ OA\,.\,OD=OB\,.\,OC$,

$\therefore$ a circle can be described round $ABCD$.

4. The tops will describe circles the centres of which are at the roots. It is evident that after collision the line joining the tops will pass through the centre of one of these circles, which it could not do before.

5. For the first part of this question see Besant's *Conics*, cap. xi. Prop. 1. Let $A$, $S$ be the vertex and focus of the given parabola. Describe a triangle $B'C'D'$ having its sides parallel to the three given straight lines. Then describe a parabola passing through the points $B'$, $C'$, $D'$, and having its axis parallel to $AS$. Let $A'$, $S'$ be the vertex and focus of this parabola. Join $S'B'$. From $S$ draw $SB$ making the angle $BSA = B'S'A'$, meeting the given parabola in $B$. Draw $BC$, $BD$ parallel to $B'C'$, $B'D'$, and join $CD$. Then by symmetry $CD$ is parallel to $C'D'$, and the triangle $BCD$ is described as required.

6. The only forces acting on the rod $AD$ are the reactions at $A$ and $D$, and when there is equilibrium these must act along $AD$. Let the force at $A$ along $AD$ be denoted by $R$. Then the rod $AB$ is acted on by $P$ and $R$ at $A$, and the forces at $B$. $\therefore$ taking moments about $B$, we have $P \cdot \sin BAC = R \cdot \sin BAD$.

Similar from $CD$ we get $Q \cdot \sin BDC = R \cdot \sin CDA$

$$\therefore \frac{P}{Q} = \frac{\sin BDC}{\sin ACD} \cdot \frac{\sin ABD}{\sin BAC} \cdot \frac{\sin BAD}{\sin ABD} \cdot \frac{\sin ACD}{\sin CDA}$$

$$= \frac{OC}{OD} \cdot \frac{OA}{OB} \cdot \frac{BD}{AD} \cdot \frac{AD}{AC},$$

$$\therefore \frac{P \cdot AC}{Q \cdot BD} = \frac{AO \cdot OC}{BO \cdot OD}.$$

7. Tripos 1878. Wednesday morning. No. 13.

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## Paper LXVIII.

1. We see that the given expression is symmetrical, and will not be altered if we put $x$ for $a$, and $a$ for $x$, and similarly for the other letters. Consider each term of the first part $(a^3 + b^3 + c^3)xyz$. Since the given expression is symmetrical, the factors of $a^3xyz$ must be $ax \cdot ay \cdot az$. Similarly the factors of $b^3xyz$ are $bx \cdot by \cdot bz$, and for $c^3xyz$ are $cx \cdot cy \cdot cz$. Now from the symmetry of the expression we cannot have $ax$ and $bx$ in the same factor. $\therefore$ the required expression is

$$(ax + by + cz)(ay + bz + cx)(az + bx + cy).$$

2. By Fermat

$$(np)^{m-1} - 1 = M(m); \quad (pm)^{n-1} - 1 = M(n); \quad (mn)^{p-1} - 1 = M(p)$$

$$\therefore \{(np)^{m-1} - 1\}\{(pm)^{n-1} - 1\}\{(mn)^{p-1} - 1\} = M(mnp)$$

$$\therefore m^{p+n-2} \cdot p^{m+n-2} \cdot n^{m+p-2}$$

$$- (n^{m-1}p^{m+n-2} \cdot m^{n-1} + m^{p-1} \cdot n^{m+p-2} \cdot p^{m-1}$$

$$+ m^{n+p-2} \cdot n^{p-1} \cdot p^{n-1})$$

$$+ (np)^{m-1} + (pm)^{n-1} + (mn)^{p-1} - 1 = M(mnp).$$

Now the first part of the expression $= M(mnp)$ by inspection

$$\therefore (np)^{m-1} + (pm)^{n-1} + (mn)^{p-1} - 1 = M(mnp).$$

3. Let $ABC$ be the given triangle, $D$, $E$, $F$ the middle points of $BC$, $CA$, $AB$. Then if $P$ be the centre of the circle touching the given circles, the points of contact are evidently on $PD$, $PE$, $PF$, produced.

$$\therefore D = 2PD + a = 2PE + b = 2PC + c.$$

$$\therefore 2D = 2(PE + PF) + b + c.$$

$$\therefore PE + PF = D - \frac{1}{2}(2s - a) = D - s + \frac{a}{2}.$$

$$\therefore PE + PF + EF = D - s + a,$$

$$\therefore \text{if } 2\sigma_1 \text{ be perimeter of } PEF,\ \sigma_1 = \frac{D - s + a}{2}.$$

Now $\Delta DEF = PEF + PFD + PDE$ . . . . . (1)

$$\text{and } DEF = \sqrt{\frac{s}{2} \cdot \frac{s-a}{2} \cdot \frac{s-b}{2} \cdot \frac{s-c}{2}},$$

$$PEF = \sqrt{\frac{D-s+a}{2} \cdot \frac{D-s}{2} \cdot \frac{s-c}{2} \cdot \frac{s-b}{2}},$$

$$PFD = \sqrt{\frac{D-s+b}{2} \cdot \frac{D-s}{2} \cdot \frac{s-c}{2} \cdot \frac{s-a}{2}},$$

$$PDE = \sqrt{\frac{D-s+c}{2} \cdot \frac{D-s}{2} \cdot \frac{s-a}{2} \cdot \frac{s-b}{2}},$$

$\therefore$ substituting in (1) and dividing both sides by

$$\sqrt{\frac{D-s}{2} \cdot \frac{s-a}{2} \cdot \frac{s-b}{2} \cdot \frac{s-c}{2}}$$

we have

$$\sqrt{\frac{D}{s-a}-1}+\sqrt{\frac{D}{s-b}-1}+\sqrt{\frac{D}{s-c}-1}=\sqrt{\frac{s}{D-s}}$$

4. (1) The three lines are evidently diameters of the nine-point circle of the triangle $ABC$.

(2) Let $d, e, f$ be the middle points of $AD$, $BE$, $CF$.

Then $2dE' = DC$, $2eF' = AE$, $2fE' = AF$,

$\therefore$ for the triangle $D'E'F'$ we have

$$dE' \cdot eF' \cdot fD' = fE' \cdot dF' \cdot eD'.$$

$\therefore$ $D'd$, $E'e$, $F'f$ meet in a point $K$.

For further information respecting $K$, see an article by Mr. Tucker in the *Quarterly Journal of Pure and Applied Mathematics*, Vol. xx., No. 78.

It can also be proved that $K$ is the point determined in XXXIX. No. 5.

5. Let $PY$ meet the axis in $T$, and draw the ordinates $PN$, $QM$.

Then by Bes. *Con.* Cap. II. Prop xxi., $AN \cdot AM = AR^2$.

From similar triangles $AQM$, $PTN$,

$$QM^2 : AM^2 :: NT^2 : PN^2,$$

$$\therefore 4AS \cdot AM : AM^2 :: 4AN^2 : 4AS \cdot AN,$$

$$\therefore 4AS : AM :: AN : AS,$$

$$\therefore 4AS^2 = AM \cdot AN = AR^2. \quad \therefore AR = 2AS.$$

6. Let $D$, $E$, $F$ be the middle points of $BC$, $CA$, $AB$, and let $O$ be the centre of gravity of $ABC$. Then the forces represented by $PB$ and $PC$ have a resultant represented by $2PD$; and forces represented by $2PD$ and $PA$ have a resultant represented by $3PO$, since $AO = 2OD$. $\therefore$ if the resultant is constant, the locus of $P$ is a circle whose centre is $O$. If there is equilibrium the resultant vanishes, and $P$ coincides with $O$

7. Draw $PM$ perpendicular to $AB$.

The time of flight from $A$ to $B$ is $\frac{2u \sin \alpha}{g}$. $\therefore t' = \frac{2u \sin \alpha}{g} - t$,

$$\therefore \tfrac{1}{2}gtt' = u \sin \alpha \cdot t - \tfrac{1}{2}gt^2 = PM.$$

## Paper LXIX.

1. Let $a$ denote the number.

Then $\frac{x}{y} = \frac{120}{a - 120}$; $\frac{x}{z} = \frac{140}{a - 140}$; $\frac{y}{z} = \frac{126}{a - 126}$,

$$\therefore \frac{140(a - 126)}{126(a - 140)} = \frac{x}{y} = \frac{120}{a - 120},$$

$$\therefore (a - 126)(a - 120) = 108(a - 140),$$

$$\therefore a^2 - 354a - 30240 = 0,$$

$$\therefore (a - 210)(a - 144) = 0.$$

$\therefore a = 210$ or 144, both of which numbers will be found to satisfy the given conditions.

2. (1) Expand and simplify, and we get

$$3x^2 - 2969x - 31000 = 0,$$

$$\therefore x = \tfrac{1}{6}\left\{2969 \pm \sqrt{8814961 + 372000}\right\}$$

$$= \tfrac{1}{6}(2969 \pm 3031) = 1000, \text{ or } -\frac{31}{3}.$$

(2) From (1) $a(2x + y + z) = (x + y + z)^2 - x^2$

$$= (2x + y + z)(y + z),$$

$$\therefore (2x + y + z)(y + z - a) = 0.$$

$\therefore$ either $2x + y + z = 0$; (4) or $y + z - a = 0$; (5)

Similarly $x + 2y + z = 0$; (6) or $z + x - b = 0$; (7)

and $x + y + 2z = 0$; (8) or $x + y - c = 0$; (9)

From these we can obtain 8 suppositions by taking together the following sets. 4, 6, 8; 5, 7, 9; 4, 7, 8; 6, 9, 4; 8, 5, 6; 5, 6, 9; 7, 8, 5; 9, 4, 7.

These are all simple equations of the first degree, and give us the following solutions

$x = 0$; $y = 0$; $z = 0$;

$x = \frac{1}{2}(-a + b + c)$; $y = \frac{1}{2}(a - b + c)$; $z = \frac{1}{2}(a + b - c)$;

$$x = \frac{b}{2}; \qquad y = -\frac{3b}{2}; \qquad z = \frac{b}{2};$$

$$x = \frac{c}{2}; \qquad y = \frac{c}{2}; \qquad z = -\frac{3c}{2};$$

$$x = -\frac{3a}{2}; \qquad y = \frac{a}{2}; \qquad z = \frac{a}{2};$$

It will be seen on trial that the sets 5, 6, 9; 7, 8, 5; 9, 4, 7 are inconsistent.

(3) Multiply (1), (2), (3) respectively first by $y$, $z$, $x$, and then by $z$, $x$, $y$, and add. We get

$$ay + bz + cx = 0$$
$$az + bx + cy = 0,$$

$$\therefore \frac{x}{bc - a^2} = \frac{y}{ca - b^2} = \frac{z}{ab - c^2} = \lambda \text{ suppose.}$$

$$\therefore a = \lambda^2 \{(ca - b^2)(ab - c^2) - (bc - a^2)^2\}$$
$$= \lambda^2 a(3abc - a^3 - b^3 - c^3).$$

This gives $\lambda$, and $\therefore$ $x$, $y$, $z$ are known.

3. Let $S$ be the centre of the circle, $Y$ the middle point of $PQ$, and from $Y$ draw $YA$ perpendicular to $OS$.

Then $\quad \frac{1}{2}(OP^2 + OQ^2) = OY^2 + PY^2 = \text{const.}$

and $\quad SY^2 + PY^2 = SP^2 = \text{const.}$

$$\therefore OY^2 - SY^2 = \text{const.}$$

$\therefore$ the locus of $Y$ is the straight line $YA$.

Now since $S$ is a fixed point, and $Y$ always lies on the straight line $YA$, and $SYP$ is a right angle, $\therefore$ $YP$ always touches a parabola whose focus is $S$, and $AY$ is the tangent at the vertex $A$.

4. $\quad e^{\alpha+\beta i} = \sin(\theta + \phi i) = \frac{1}{2i}(e^{\theta i - \phi} - e^{-\theta + \phi})$

$$\therefore 2i \cdot e^{\alpha}(\cos\beta + i\sin\beta) = e^{-\phi}(\cos\theta + i\sin\theta) - e^{\phi}(\cos\theta - i\sin\theta)$$

$\therefore$ equating real and unreal parts

$$\cos\theta(e^{\phi} - e^{-\phi}) = 2e^{\alpha} \cdot \sin\beta; \ \sin\theta(e^{\phi} + e^{-\phi}) = 2e^{\alpha} \cdot \cos\beta,$$

$$\therefore\ 4e^{2\alpha} = \cos^2\theta(e^{2\phi} + e^{-2\phi} - 2) + \sin^2\theta(e^{2\phi} + e^{-2\phi} + 2)$$
$$= e^{2\phi} + e^{-2\phi} - 2\cos 2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

Also $\dfrac{\cos\theta\cos\beta}{\sin\theta\sin\beta} = \dfrac{e^{\phi} + e^{-\phi}}{e^{\phi} - e^{-\phi}}$, $\therefore\ \dfrac{\cos(\theta - \beta)}{\cos(\theta + \beta)} = \dfrac{2e^{\phi}}{2e^{-\phi}} = e^{2\phi}$,

$$\therefore\ \cos(\theta - \beta) = e^{2\phi}\cos(\theta + \beta).$$

5. Let $FD$ meet the tangent at $A$ to the circle $ADC$ in $G$.

Then the angle $DAG = ACD = DAC$

and the angle $ADG = ACF = ACD + DCF = DAC + DAF$
$= DAC + \frac{1}{2}\,DAC.$

$\therefore\ GAD + ADG = 2DAC + \frac{1}{2}DAC = ABC + \frac{1}{2}BAC =$ a right angle.

$\therefore\ DGA$ is a right angle.

Similarly it may be shewn for $FE$.

6. Considering the motion normal to the plane, the times of flight between the bounds are

$$\frac{2v}{g\cos a},\ \frac{2ev}{g\cos a},\ \frac{2e^2v}{g\cos a},\quad . \quad . \quad .$$

$$\therefore\ \text{the whole time} = \frac{2v}{g\cos a}(1 + e + e^2 + \ . \ . \ .)$$
$$= \frac{2v}{g\cos a} \cdot \frac{1}{1 - e}.$$

Considering the motion parallel to the plane, since the particle is acted on by a constant accelerating force $g\sin a$, and starts with no initial velocity parallel to the plane,

$$\therefore\ S = \tfrac{1}{2}g\sin a \,.\, t^2.$$
$$= \tfrac{1}{2} \,.\, g\sin a \,.\, \frac{4v^2}{g^2\cos^2 a} \cdot \frac{1}{(1 - e)^2}$$
$$= \frac{2v^2\sin a}{g\cos^2 a} \cdot \frac{1}{(1 - e)^2}.$$

7. The equation to the normal at any point can be written,

$$x \,.\, \frac{a^2}{x'(a^2 - b^2)} - y\,\frac{b^2}{y'(a^2 - b^2)} = 1 \quad . \quad . \quad . \quad . \quad (A).$$

If $(\xi, \eta)$ be the coordinates of the point, the equation to its polar is

$$\frac{x\xi}{a^2} + \frac{y\eta}{b^2} = 1 \quad . \quad . \quad . \quad . \quad (B).$$

If this is a normal chord, by comparing $(B)$ with $(A)$ we have

$$\frac{\xi}{a^2} = \frac{a^2}{x'(a^2 - b^2)}; \quad \frac{\eta}{b^2} = -\frac{b^2}{y'(a^2 - b^2)},$$

$$\therefore \frac{x'}{a} = \frac{a^3}{\xi(a^2 - b^2)}, \quad \frac{y'}{b} = -\frac{b^3}{\eta(a^2 - b^2)}$$

$$\therefore 1 = \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = \frac{a^6}{\xi^2(a^2 - b^2)^2} + \frac{b^6}{\eta^2(a^2 - b^2)^2}$$

$$\therefore \frac{a^6}{\xi^2} + \frac{b^6}{\eta^2} = (a^2 - b^2)^2 = a^4e^4.$$

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## Paper LXX.

1. (1) By inspection $x = 1$ satisfies the equation. $\therefore$ $x = 1$ is a root. The product of the roots is $\frac{c(a - b)}{a(b - c)}$. $\therefore$ the other root is $\frac{c(a - b)}{a(b - c)}$.

(2) Arrange the equation thus:

$$(x^4 - \tfrac{1}{4}) - x\sqrt{2}\sqrt{x^4 - \tfrac{1}{4}} + \frac{x^2}{2} = \frac{x^2 - 1}{2}.$$

Extract square root. $\therefore \sqrt{x^4 - \tfrac{1}{4}} - \frac{x}{\sqrt{2}} = \pm \frac{\sqrt{x^2 - 1}}{\sqrt{2}}$,

$$\therefore \sqrt{x^4 - \tfrac{1}{4}} = \frac{x \pm \sqrt{x^2 - 1}}{\sqrt{2}},$$

$$\therefore 4x^4 - 1 = 4x^2 - 2 \pm 4x\sqrt{x^2 - 1}.$$

$$\therefore 4x^2(x^2 - 1) \pm 4x\sqrt{x^2 - 1} + 1 = 0.$$

Extract square root $\therefore 2x\sqrt{x^2-1} \pm 1 = 0, \quad \therefore 4x^4 - 4x^2 = 1,$

$$\therefore x^2 = \frac{2 \pm \sqrt{2}}{2}. \quad \therefore x = \pm \sqrt{\frac{2 \pm \sqrt{2}}{2}}.$$

(3) By inspection $x = a$ is a root. The equation is a quadratic of the form $x^2 + Ax + a^3 = 0$. $\therefore$ the second root is $a^2$.

2. Tripos 1875. Wednesday morning. No. 3.

3. Let $ABCD$ be a parallelogram, $E$ the intersection of $AC$ and $BD$. Describe circles round $AEB$, $CED$. These are evidently equal. Let $O$, $O'$ be their centres, $F$, $G$ the middle points of $AB$, $CD$. Then $FG$ passes through $E$, and the triangles $OEF$, $O'EG$ are equal in all respects,

$\therefore$ the angle $FEO = GEO'$. Add to each $GEO$.

$\therefore OEG + GEO' = OEF + OEG = 2$ right angles.

$\therefore OO'$ passes through $E$, *i.e.* the line joining the centres passes through a common point. $\therefore$ the circles touch at $E$.

Consider the circles round $AED$, $AEB$. Let the tangent at $E$ to the former cut $AB$ in $H$, and the tangent to the latter at $E$ cut $BC$ in $K$.

Then the angle $HEB = DAE$, and $BEK = HAE$, $\therefore HEK = DAB$.

4. $BD = s - b, \quad \therefore AD^2 = c^2 + (s-b)^2 - 2c(s-b)\cos B.$

$CD = s - c, \quad \therefore AD^2 = b^2 + (s-c)^2 - 2b(s-c)\cos B.$

$$\therefore 2AD^2 = b^2 + c^2 + (s-b)^2 + (s-c)^2 - 2s(c\cos B + b\cos C) + 2bc(\cos B + \cos C)$$
$$= 2\{b^2 + c^2 + s^2 - 2s^2 + bc(\cos A + \cos B\cos C) - bc\cos A\},$$

$$\therefore AD^2 = b^2 + c^2 - s^2 + bc(\cos A + \cos B + \cos C) - \frac{b^2 + c^2 - a^2}{2}$$

$$= \frac{a^2 + b^2 + c^2}{2} - s^2 + bc(\cos A + \cos B + \cos C).$$

So $$BE^2 = \frac{a^2 + b^2 + c^2}{2} - s^2 + ca(\cos A + \cos B + \cos C),$$

$$CF^2 = \frac{a^2 + b^2 + c^2}{2} - s^2 + ab(\cos A + \cos B + \cos C),$$

$\therefore$ if $AD^2 + CF^2 = 2BE^2$, we have at once

$$ab + bc = 2ca, \quad \therefore b = \frac{2ac}{a+c}, \ \therefore a, b, c \text{ are in H.P.}$$

5. Let the four tangents form four triangles whose vertices are $A, B, C, D, E, F$, so that $A, B, C$; $B, F, E$; $C, F, D$; are collinear points in the above order. Describe circles round any two of the triangles thus formed, and let them intersect in $S$. Then $S$ is the focus. From $S$ draw $SG$, $SH$ perpendicular to $AB$ and $AD$, and join $GH$, and draw $SK$ at right angles to $GH$. Then $GH$ is the tangent at the vertex, and $K$ is the vertex. Through $S$ draw $LSM$ at right angles to $SA$, making $SL = SM = 2AS$.

To find the point of contact of $AB$, join $GS$, and make the angle $GSN = GSA$, and let $SN$ meet $AB$ in $N$. Then $N$ is the point required. In a similar manner we can find the points of contact of the other three tangents.

6. Let $P$, $Q$ be the two particles, $\alpha$, $\alpha'$ the inclinations of the planes on which they are. Suppose $P$ on the point of slipping up, and $Q$ on the point of slipping down. Let $R$ and $R'$ denote the whole resistances on $P$ and $Q$ when motion is about to ensue. Then if $\lambda$ be the angle of friction, $R$ and $R'$ make each the angle $\lambda$ with the normals at $P$ and $Q$.

Now since the tensions of the string at $P$ and $Q$ are equal, and their directions equally inclined to the vertical, and the weights of $P$ and $Q$ equal, we must have $R$ and $R'$ equal and equally inclined to the horizon (or vertical).

$$\therefore \alpha + \lambda = \alpha' - \lambda, \quad \therefore \alpha' - \alpha = 2\lambda.$$

7. Since the force of gravity is counteracted by the impressed force, the body will move as though under the action of no forces, and $\therefore$ will move in a straight line directly from the point of projection with the velocity of projection, which velocity has carried it over the distance which it has travelled.

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## Paper LXXI.

1. (1) $$a - b = (x + a)\sqrt{\frac{x^2 + b^2}{x^2 + a^2}} - (x + b)\sqrt{\frac{x^2 + a^2}{x^2 + b^2}},$$

$$\therefore (a-b^2) = \frac{(x+a)^2(x^2+b^2)}{x^2+a^2} + \frac{(x+b)^2(x^2+a^2)}{x^2+b^2} - 2(x+a)(x+b),$$

$$\therefore a^2 + b^2 + 2x^2 = \frac{(x+a)^2(x^2+b^2)}{x^2+a^2} + \frac{(x+b)^2(x^2+a^2)}{x^2+b^2} - 2(a+b)x,$$

$$\therefore (x+a)^2+(x+b)^2=\frac{(x+a)^2(x^2+b^2)}{x^2+a^2}+\frac{(x+b)^2(x^2+a^2)}{x^2+b^2},$$

$$\therefore \frac{1}{(x+a)^2}+\frac{1}{(x+b)^2}=\frac{x^2+b^2}{x^2+a^2}\cdot\frac{1}{(x+b)^2}+\frac{x^2+a^2}{x^2+b^2}\cdot\frac{1}{(x+a)^2},$$

$$\therefore \frac{1}{(x+a)^2}-\frac{x^2+a^2}{x^2+b^2}\cdot\frac{1}{(x+a)^2}=\frac{x^2+b^2}{x^2+a^2}\cdot\frac{1}{(x+b)^2}-\frac{1}{(x+b)^2},$$

$$\therefore \frac{x^2+b^2-x^2-a^2}{(x+a)^2(x^2+b^2)}=\frac{x^2+b^2-x^2-a^2}{(x+b)^2(x^2+a^2)},$$

$$\therefore (x^2+a^2)(x+b)^2=(x^2+b^2)(x+a)^2.$$

$\therefore x=0$ or $\pm\sqrt{ab}$. The value $x=0$ is inadmissible, but will satisfy the equation obtained by changing the signs of the radicals.

(2) Assume $x=b\tan\theta$, $y=a\tan\phi$. Then the equations may be written

$$\frac{b}{a}\sec^2\theta=\sqrt{2}(\tan\theta+\tan\phi);\quad \frac{a}{b}\sec^2\phi=\sqrt{2}(\tan\theta\tan\phi-1).$$

$$\therefore \sec^2\theta\sec^2\phi=-2\frac{\sin(\theta+\phi)\cos(\theta+\phi)}{\cos^2\theta\cos^2\phi},$$

$$\therefore \sin 2(\theta+\phi)=-1,\quad \therefore \theta+\phi=n\pi-\frac{\pi}{4}.$$

$$\frac{b}{a}\sec^2\theta=\sqrt{2}\left\{\tan\theta-\tan\left(\frac{\pi}{4}+\theta\right)\right\}$$

$$=\sqrt{2}\left\{\tan\theta-\frac{1+\tan\theta}{1-\tan\theta}\right\}=-\sqrt{2}\cdot\frac{1+\tan^2\theta}{1-\tan\theta},$$

$$\therefore \frac{b}{a}(\tan\theta-1)=\sqrt{2};\quad \therefore x=b\tan\theta=b+a\sqrt{2}.$$

$$\therefore y=a\tan\phi=-a\tan\left(\frac{\pi}{4}+\theta\right)=a\frac{\tan\theta+1}{\tan\theta-1}=a+b\sqrt{2}$$

An algebraical solution will be found at the end of Todh. *Alg.*

(3) For solution see end of volume.

2. The number of ways is evidently the number of combinations of $3m$ things taken $2m$ at a time, when there are $m$ alike of one set, $m$ alike of another set, and $m$ alike of a third set.

∴ Todh. *Alg.* Art. 811, the required number
= coefficient of $x^{2m}$ in the expansion of

$$\frac{1-x^{m+1}}{1-x}\cdot\frac{(1-x)^{m+1}}{1-x}\cdot\frac{(1-x)^{m+1}}{1-x}$$

$$= \text{coefficient of } x^m \text{ in } \frac{(1-x^{m+1})^3}{(1-x)^3}.$$

Now this expression

$$= (1-x^{m+1})^3\left\{1+3x+\ \ldots+\frac{(m+1)(m+2)}{\lfloor 2}x^m+\ldots..\right\}$$

and coefficient of $x^m$ is evidently $\frac{1}{2}(m+1)(m+2)$.

3. $\cos B \sin\frac{A+B}{2}\sin\frac{C-D}{2}$

$$= \tfrac{1}{2}\sin\frac{C-D}{2}\left\{\sin\frac{A+3B}{2}+\sin\frac{A-B}{2}\right\}$$

$$= \tfrac{1}{4}\left\{\cos\frac{A+3B-C+D}{2}-\cos\frac{A+3B+C+D}{2}\right.$$

$$\left.+\cos\frac{A-B-C+D}{2}-\cos\frac{A-B+C-D}{2}\right\}.$$

By symmetric changes of the letters $B, C, D$ we can write down the other two terms of the given expression, which on addition we find to be

$$= \tfrac{1}{4}\left\{\left(\cos\frac{A+3B-C+D}{2}-\cos\frac{A-B+3C+D}{2}\right)\right.$$

$$+\left(\cos\frac{A+B+3C-D}{2}-\cos\frac{A+B-C+3D}{2}\right)$$

$$\left.+\left(\cos\frac{A-B+C+3D}{2}-\cos\frac{A+3B+C-D}{2}\right)\right\}$$

$$= -\tfrac{1}{2}\sin\frac{A+B+C+D}{2}\left\{\sin(B-C)+\sin(C-D)+\sin(D-B)\right\}$$

$$= 2\sin\frac{A+B+C+D}{2}\sin\frac{B-C}{2}\sin\frac{C-D}{2}\sin\frac{D-B}{2}.$$

Todh. *Trig.* Cap. viii. Ex. 3.

4. Add $4 + 5\tan^2\theta$ to each side of the given equation,

$$\therefore 12\tan^2\theta + 8\sqrt{3}\tan\theta + 4 = 5(1 + \tan^2\theta) = 5\sec^2\theta,$$

$$\therefore 3\tan^2\theta + 2\sqrt{3}\tan\theta + 1 = \tfrac{5}{4}\sec^2\theta,$$

$$\therefore \sqrt{3}\tan\theta + 1 = \pm\frac{\sqrt{5}}{2}\sec\theta.$$

$$\therefore \frac{\sqrt{3}}{2}\sin\theta + \tfrac{1}{2}\cos\theta = \pm\frac{\sqrt{5}}{4}; \quad \therefore \sin\left(\theta + \frac{\pi}{6}\right) = \pm\frac{\sqrt{5}}{4}.$$

From the given condition we must take the positive sign.

$$\therefore L\sin\left(\theta + \frac{\pi}{6}\right) = 10 + \log\frac{\sqrt{5}}{4} = 10 + \tfrac{1}{2}\log 5 - \log 4$$

$$= 10 + \tfrac{1}{2}\log 10 - 2\log 2$$

$$= 10 + \tfrac{1}{2} - \tfrac{5}{2}\log 2$$

$$= 9\,.\,7474250.$$

Dif. of 60″ in angle gives dif. in log of ·0001874,

$$\therefore 1874 : 507 :: 60'' : 16''\,.\,23$$

$$\therefore \theta + \frac{\pi}{6} = 33^\circ\,.\,59'\,.\,16''\,.\,23$$

$$\therefore \theta = 3^\circ\,.\,59'\,.\,16''\,.\,2 \text{ nearly.}$$

5. Let $ABCD$ be the first tetrahedron. Let $E$, $F$ be the middle points of $BD$, $CD$, and $P$, $Q$ the centres of gravity of the faces $ABD$, $ACD$.

Then $PQ : EF :: AP : AE :: 2 : 3$, $\therefore PQ = \frac{2}{3}EF = \frac{1}{3}BC$.

$\therefore$ volume of 1st tetrahedron : volume of 2nd $:: BC^3 : PQ^3 :: 27 : 1$.

Let $a$ be a side of the first cube, $A'$, $B'$ the middle points of two adjacent sides.

Then $A'B'^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2 = \frac{a^2}{2}$.

Now let $O$ be a vertex of the octohedron, corresponding to $A'B'$, and let $P'$, $Q'$ be 2 corners of the 2nd cube.

Then $P'Q' : E'F' :: OP : OE :: 2 : 3$, $\therefore P'Q' = \frac{2}{3}E'F' = \frac{1}{3}B'C'$.

And $B'C'^2 = 2B'D'^2 = \frac{2a^2}{2} = a^2$; $\therefore P'Q' = \frac{a}{3}$.

$\therefore$ volume of 1st cube : volume of 2nd :: $a^3 : P'Q'^3$ :: 27 : 1.

Now volume of 1st cube = volume of 1st tetrahedron,

$\therefore$ volume of 2nd " = " 2nd "

6. Project the conic into a circle, and denote corresponding points in the circle by small letters. Then *oe*, *od* are at right angles, as are also *oc* and *de*. $\therefore$ *eb* . *bd* = $bo^2$ = const. $\therefore$ also $EB \cdot BD$ = const. $\therefore$ if the circle round *EOD* meet *OC* in *F*, $OB \cdot BF = EB \cdot BD$ = const. $\therefore$ *F* is a fixed point.

* 7. Let *E* be the point of projection. Since the particles have equal velocities, the parabolas described for different directions of projection have all a common directrix *KK'*, such that if *EM* be the perpendicular upon it from *E*, the velocity of projection is equal to that due to a height *EM*.

Let *ET* be any direction of projection; *S*, *A* the focus and vertex of the parabola described. Then if *SY* be perpendicular on the tangent *ET*, *SY* passes through *M*, and $SY = YM$.

Now since *E* and *M* are fixed points, the locus of *Y* is the circle upon *EM* as diameter; and since $AL = 2LY$, the locus of *A* is an ellipse having *EM* for its minor axis, and its axes in the ratio of 2 : 1.

In the second case, let a horizontal through *E* the point of projection be drawn at right angles to the wall, meeting it in the point *H*. Let *P* be the point where one of the particles strikes the wall, *A'* the vertex of its path after impact, and *A* the vertex of the path it would have described if there had been no wall. Draw *AF*, *A'F'* perpendicular to *EH*. Through *P* draw a line parallel to *EH*, meeting *AF*, *A'F'* in *N*, *N'*. Join *AA'* meeting *GH* in *M*. Then since the vertical motion is unaltered, $AF = A'F'$. $\therefore$ *AA'* is perpendicular to *GH*, and the time from *P* to *A* = the time from *P* to *A'*; and since the horizontal velocity is diminished in the ratio *e* : 1. $\therefore PN' = ePN$, or $A'M = e \cdot AM$.

Now since *AA'* is perpendicular to *GH*, and $A'M = eAM$, and the locus of *A* is an ellipse by the first case, $\therefore$ the locus of *A'* is also an ellipse.

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## Paper LXXII.

1. (1) If 1, $w$, $w^2$ denote the cube roots of unity, we know that

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + wb + w^2c)(a + w^2b + wc),$$

$\therefore$ the given expression is the product of three factors $X$, $Y$, $Z$, where

$$X = x^2 + y^2 + z^2 + 2yz + 2zx + 2xy = (x + y + z)^2,$$

$$Y = x^2 + 2yz + w(y^2 + 2zx) + w^2(z^2 + 2xy) = (x + w^2y + wz)^2,$$

$$Z = x^2 + 2yz + w^2(y^2 + 2zx) + w(z^2 + 2xy) = (x + wy + w^2z)^2,$$

$\therefore$ the given expression $= X \,.\, Y \,.\, Z$

$$= \{(x + y + z)(x + w^2y + wz)(x + wy + w^2z)\}^2$$

$$= (x^3 + y^3 + z^3 - 3xyz)^2.$$

(2) $bc - a^2 = \dfrac{abc}{a} - a^2$. $\therefore$ if $a$, $b$, $c$ be the roots of the equation

$$x^3 - px^2 + qx - r = 0, \quad \therefore\ q = bc + ca + ab,\ r = abc,$$

and if we put $y = \dfrac{abc}{x} - x^2$,

then
$$xy = abc - x^3 = r - x^3 = qx - px^2,$$

$$\therefore\ px + y = q, \quad \therefore\ x = \frac{q - y}{p},$$

$$\therefore\ y\left(\frac{q - y}{p}\right) = r - \left(\frac{q - y}{p}\right)^3,$$

or
$$y^3 - y^2(3q - p^2) + y \,.\, q(3q - p^2) + p^3r = 0.$$

The roots of this equation are evidently $bc - a^2$, $ca - b^2$, $ab - c^2$, and sum of the roots $= 3q - p^2$,

and sum of product of roots 2 at a time $= q(3q - p^2)$

$$= q \text{ (sum of roots)},$$

$$\therefore\ (ca - b^2)(ab - c^2) + (ab - c^2)(bc - a^2) + (bc - a^2)(ca - b^2)$$

$$= (bc + ca + ab)(bc + ca + ab - a^2 - b^2 - c^2).$$

2. $\cos x + \cos 5x + \ldots + \cos(4n - 3)x$ ($n$ terms)

$$= \cos\left(x + \frac{n - 1}{2} \,.\, 4x\right) \sin\frac{n}{2} \,.\, 4x \operatorname{cosec} 2x$$

$$= \cos(2n - 1)x \sin 2nx \operatorname{cosec} 2x,$$

$$\sin 3x + \sin 7x + \ldots + \sin(4n - 1)x \quad (n \text{ terms})$$

$$= \sin\left(3x + \frac{n-1}{2} \cdot 4x\right) \sin \frac{n}{2} \cdot 4x \operatorname{cosec} 2x$$

$$= \sin (2n + 1)x \sin 2nx \operatorname{cosec} 2x,$$

$\therefore$ the given equation may be written

$$\{\cos (2n - 1)x + \sin (2n + 1)x\} \frac{\sin 2nx}{\sin 2x} = \frac{\sin x + \cos x}{2 \sin 2x}.$$

Now $\sin 2x \neq 0$,

$$\therefore \sin x + \cos x = 2 \sin 2nx \{\cos (2n - 1)x + \sin (2n + 1)x\}$$

$$= \sin (4n - 1)x + \sin x + \cos x - \cos (4n + 1)x$$

$$\therefore \cos (4n + 1)x = \sin (4n - 1)x = \cos \left\{\frac{\pi}{2} - (4n - 1)x\right\}$$

$$\therefore (4n + 1)x = 2n\pi \pm \left\{\frac{\pi}{2} - (4n - 1)x\right\}$$

$$\therefore \text{either } 8nx = (4n + 1)\frac{\pi}{2}; \text{ or } 2x = (4n - 1)\frac{\pi}{2}.$$

3. Write the given equations in the form

$$(x \cos a + y \sin a) \cos \phi - (x \sin a - y \cos a) \sin \phi = a \sin 2\phi. \quad (1)$$

$$- (x \sin a - y \cos a) \cos \phi - (x \cos a + y \sin a) \sin \phi = 2a \cos 2\phi. \quad (2)$$

Multiply (1) by $\cos \phi$, and (2) by $\sin \phi$. and subtract.

$$\therefore x \cos a + y \sin a = a(\sin 2\phi \cos \phi - 2 \cos 2\phi \sin \phi)$$

$$= a \sin \phi (1 - \cos 2\phi)$$

$$= 2a \sin^3 \phi.$$

So $\quad x \sin a - y \cos a = 2a \cos^3 \phi,$

$$\therefore (x \cos a + y \sin a)^{\frac{2}{3}} + (x \sin a - y \cos a)^{\frac{2}{3}} = (2a)^{\frac{2}{3}}.$$

4. Let $(h, k)$ be coordinates of $A$.

Then since the equation to $CM$ is $y = \frac{b}{a} x$, the equation to $AM$ is

$$b(y - k) + a(x - h) = 0,$$

$\therefore$ coordinates of $M$ are $\frac{a}{a^2+b^2}(ah+bk)$, $\frac{b}{a^2+b^2}(ah+bk)$.

Writing $-a$ for $a$, the coordinates of $N$ are

$$-\frac{a}{a^2+b^2}(bk-ah),\quad \frac{b}{a^2+b^2}(bk-ah),$$

$\therefore$ coordinates of $O$, the middle point of $MN$ are $\frac{a^2h}{a^2+b^2}, \frac{b^2k}{a^2+b^2}$,

$\therefore$ equation to $AO$ is

$$(y-k)\left(\frac{a^2h}{a^2+b^2}-h\right)=(x-h)\left(\frac{b^2k}{a^2+b^2}-k\right)$$

or
$$b^2h(y-k)=a^2k(x-h),$$

which is evidently perpendicular to the polar of $A$ whose equation is

$$\frac{xh}{a^2}+\frac{yk}{b^2}=1.$$

* This may be also proved geometrically as follows.

Join $CA$ cutting the ellipse in $P$, and draw $PM'$, $PN'$ perpendicular to the equiconjugate diameters. Let the normal at $P$ meet $M'N'$ in $L'$, and draw $AL$ parallel to $PL'$, and $\therefore$ perpendicular to the polar of $A$ which is parallel to the tangent at $P$. Let $AL$ meet $MN$ in $L$. Then since circles will go round $PN'CM'$ and $ANCM$,

$\therefore PN'M' = PCM' = ACM = ANM$. $\therefore MN$ is parallel to $M'N'$.

$\therefore$ the triangles $AMN$, $PM'N'$ are similar, and $AL$ is parallel to $PL'$,

$$\therefore ML : M'L' :: AL : PL' :: LN : L'N'.$$

But by XXVI. No. 7, $M'L' = L'N'$, $\therefore ML = LN$. $\therefore AL$ is the diagonal of the parallelogram which has $AM$ and $AN$ for adjacent sides.

**5.** $D'$ is the point of contact of the escribed circle opposite $A$.

Since $A$ is the centre of similitude of the inscribed and this escribed circle, being the intersection of common tangents, $\therefore$ by similar triangles

$AP : AD' ::$ radius of inscribed circle : radius of escribed circle

$:: AE : AE'$, where $E'$ is point of contact of latter circle,

$$\therefore AP : PD' :: AE : EE',$$

$$\therefore AE \cdot PD' = AP \cdot EE' = AP \cdot BC.$$

6. Let $P$ and $Q$ denote the positions of the particles. Let $R$ denote the resistance of the plane at $P$, and $T$ the tension of the string. Then the particle $P$ is acted on by three forces, viz. its weight, the tension of the string, and the resistance of the plane, and when $P$ is on the point of moving, these three forces are in one plane. Let $PD$ be the trace of $R$ on the plane, $PE$ the normal to the plane, and $PF$ the line of greatest slope. Then $PR$ makes the angle of friction ($\lambda_1$) with $PE$. Let $DPF = \phi$, and let $2\theta$ be the angle between the directions of the string.

Then resolving in the plane (1) downwards, (2) horizontally, and (3) resolving vertically, we have

$$(1) \quad W_1 \sin a = T \cos\theta + R \sin\lambda_1 \cos\phi.$$

$$(2) \quad T \sin\theta = R \sin\lambda_1 \sin\phi.$$

$$(3) \quad W_1 \cos a = R \cos\lambda_1.$$

Now when the angle at the pulley is a maximum, this angle is evidently = 2 right angles. $\theta = \frac{\pi}{2}$.

$\therefore$ from (1), (2), (3) we have

$$T^2 + W_1^2 \sin^2 a = R^2 \sin^2\lambda_1 = W_1^2 \tan^2\lambda_1 \cos^2 a = W_1^2 \mu_1^2 \cos^2 a,$$

$$\therefore T^2 = W_1^2 (\cos^2 a \,.\, \mu_1^2 - \sin^2 a) = W_2^2 (\cos^2 a \,.\, \mu_2^2 - \sin^2 a) \text{ by symmetry,}$$

$$\therefore W_1^2 : W_2^2 :: \sin^2 a - \mu_2^2 \cos^2 a : \sin^2 a - \mu_1^2 \cos^2 a$$

$$:: 1 - \mu_2^2 \cot^2 a : 1 - \mu_1^2 \cot^2 a.$$

7. Consider the motion normal to the plane. The times of the several paths are

$$\frac{2v \sin a}{g}, \quad \frac{2ev \sin a}{g}, \quad \ldots \quad \frac{2e^{n-1} v \sin a}{g},$$

$$\therefore \text{ the whole time } = \frac{2v \sin a}{g} (1 + e + e^2 + \ldots + e^{n-1}),$$

$$= \frac{2v \sin a}{g} \cdot \frac{1 - e^n}{1 - e}.$$

Since the horizontal component remains unaltered, and $= v \cos a$, the space described

$$= v \cos a \,.\, \frac{2v \sin a}{g} \cdot \frac{1 - e^n}{1 - e}.$$

$$= \frac{v^2 \sin 2a}{g} \cdot \frac{1 - e^n}{1 - e}.$$

## Paper LXXIII.

1. Put $a = \frac{1}{a'}$, $b = \frac{1}{b'}$, $c = \frac{1}{c'}$, ...

$$\therefore P = \cfrac{1}{1 + \cfrac{a'}{1 + \cfrac{b'}{1 + \cfrac{c'}{1 + \ldots}}}} \qquad Q = \cfrac{ab'}{1 + \cfrac{c'}{1 + \cfrac{d'}{1 + \ldots}}}$$

$$\therefore P = \cfrac{1}{1 + \cfrac{a'}{1 + \cfrac{Q}{a}}} \qquad \therefore P(1 + a + Q) = Q + a.$$

2. Let $a$ be the greatest, $c$ the least side.

Let $b = c + \gamma$, $a = b + \beta$, where $\beta < c$. $\therefore a = c + \beta + \gamma$.

$$\therefore \tfrac{2}{3}(a + b + c)(a^2 + b^2 + c^2) - (a^3 + b^3 + c^3 + 3abc)$$

$$= \tfrac{1}{3}(c - \beta)(\beta\gamma + \beta^2) + \tfrac{1}{3}c\gamma^2 + \tfrac{2}{3}\gamma^3 + \beta\gamma^2.$$

Now $c > \beta$, and $\beta, \gamma$ are both positive quantities, $\therefore$ the last expression is positive.

3. Let $O$ be the centre of the inscribed circle.

Then $EDF = \frac{1}{2}EOF = \frac{1}{2}\left(\pi - \frac{B}{2} - \frac{C}{2}\right) = \frac{1}{2}\left(\frac{\pi}{2} + \frac{A}{2}\right) = \frac{1}{4}(\pi + A)$.

Similarly $DEF = \frac{1}{4}(\pi + B)$; $EFD = \frac{1}{4}(\pi + C)$.

4. Let the common tangents intersect in $A$, and let $S, S_1, S_2$ be the points of contact in the order $A, S, S_1, S_2$. Let the circles on which are $S_1, S_2$ intersect in $E$. Join $AE$ cutting the circle $S$ in $B$ and $C$ and $S_1$ in $D$. Then since $A$ is a centre of similitude, $\therefore BS$ and $DS_1$ are parallel, as also $CS$, $ES_1$, and $DS_1$, $ES_2$.

$$\therefore EB = SS_2 \cdot \frac{AB}{AS}, \quad EC = SS_1 \cdot \frac{AC}{AS}, \quad AB \cdot AC = AS^2,$$

$\therefore$ (tangent)$^2$ from $E$ to circle $S = EB \cdot EC = SS_1 \cdot SS_2$

5. $$\frac{PR}{QN} = \frac{PK}{QK} = \frac{SP}{SQ} = \frac{PM}{QN} \quad \therefore PR = PM.$$

Similarly if $NQ$ meet $KM$ in $R'$,

$$\frac{QR'}{PM} = \frac{QK}{PK} = \frac{SQ}{SP} = \frac{QN}{PM} \quad \therefore QR' = QN.$$

6. Let $y = a$ be the equation to the straight line, and the equation to any equilateral hyperbola be of the form

$$x^2 - y^2 + Axy = B.$$

The normal at any point $(x'y')$ is

$$y - y' = \frac{Ax' - 2y'}{2x' + Ay'}(x - x').$$

If this normal coincides with the line $y = a$, we must have

$$Ax' - 2y' = 0; \quad \text{and} \quad y' = a. \quad \therefore A = \frac{2a}{x'}$$

Since $(x', a)$ is a point on the hyperbola,

$$\therefore B = x'^2 + a^2 = \frac{4a^2}{A^2} + a^2,$$

$\therefore$ the equation to any of the equilateral hyperbolas may be written

$$A^2xy + A^2(x^2 - y^2 - a^2) - 4a^2 = 0.$$

To find the envelope we must express the condition that two of the roots of this equation are equal. Let the roots be $a_1$, $a_1$, $a_3$.

Then
$$2a_1 + a_3 = -\frac{x^2 - y^2 - a^2}{xy} \quad \ldots\ldots \quad (1)$$

$$a_1^2 + 2a_1 a_3 = 0 \quad \ldots\ldots \quad (2)$$

$$a_1^2 a_3 = \frac{4a^2}{xy} \quad \ldots\ldots \quad (3)$$

From (2) $a_1(a_1 + 2a_3) = 0$, and $a_1 \neq 0$. $\therefore a_1 = -2a_3$,

$\therefore$ from (1)
$$3a_3 = \frac{x^2 - y^2 - a^2}{xy},$$

$\therefore$ from (3)
$$\left(\frac{a^2}{xy}\right)^{\frac{1}{3}} = a_3 = \frac{x^2 - y^2 - a^2}{3xy},$$

$$\therefore x^2 - y^2 - a^2 = 3(axy)^{\frac{2}{3}}.$$

7. $B = mv = m.\frac{s}{t}$; $F = mf = 2m.\frac{s}{t^2}$. $\therefore \frac{2B}{t} = F$. $\therefore 2B = Ft$.

$$2B^2 = \frac{2m^2s^2}{t^2}; \quad mFs = \frac{2m^2s^2}{t^2} = 2B^2.$$

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## Paper LXXIV.

1. By multiplication we have

$$a^2b^2c^2 = abcxyz + abx^2y^2 + acx^2z^2 + bcy^2z^2 + ax^3yz + bxy^3z + cxyz^3 + x^2y^2z^2$$

Since $xyz = abc$, this reduces to

$$a^2b^2c^2 + abx^2y^2 + bcy^2z^2 + caz^2x^2 + (ax^2 + by^2 + cz^2)\,abc = 0 \quad . \qquad (A)$$

Square the first equation.

$$\therefore a^2x^2 + y^2z^2 + 2axyz = b^2c^2; \quad \therefore a^2x^2 + y^2z^2 = b^2c^2 - 2a^2bc,$$

$\therefore$ writing $(A)$ in the form

$$a^2b^2c^2 + bc(a^2x^2 + y^2z^2) + ca(b^2y^2 + z^2x^2) + ab(c^2z^2 + x^2y^2) = 0$$

$$a^2b^2c^2 + bc(b^2c^2 - 2a^2bc) + ca(c^2a^2 - 2ab^2c) + ab(a^2b^2 - 2abc^2) = 0,$$

$$\therefore b^3c^3 + c^3a^3 + a^3b^3 = 7a^2b^2c^2.$$

2. The number in the scale of 7 has no digit higher than 6, and all its digits are even by the question, $\therefore$ they are 2, 4, 6, and those in the scale of 10 are 1, 2, 3. Let $x$, $y$, $z$ he the digits in the scale of 10, then $2x$, $2y$, $2z$ are the corresponding digits in the scale of 7.

$$\therefore x + 10y + 10^2z = 2x + 7 \,.\, 2y + 7^2 \,.\, 2z,$$

$$\therefore x = 2(z - 2y).$$

$\therefore x$ is even, and $z > 2y$. Now $x$, $y$, $z$ can only be selected from the digits 1, 2, 3. $\therefore x = 2$, $y = 1$, $z = 3$.

$\therefore$ the number is 312 in scale of 10, and 624 in the scale of 7.

3. $$p(\cos^4\theta - \sin^4\theta) - q(\cos^4\phi - \sin^4\phi) = q - p,$$

$$\therefore p\cos 2\theta - q\cos 2\phi = q - p,$$

$$\therefore p\cos^2\theta = q\cos^2\phi.$$

Now $q = p\cos^4\theta - q\cos^4\phi = p\cos^4\theta - \frac{p^2}{q}\cos^4\theta.$

$$\therefore \cos\theta = \sqrt[4]{\frac{q^2}{p(q-p)}}.$$

And $q = \frac{q^2}{p}\cos^4\phi - q\cos^4\phi.$ $\therefore \cos\phi = \sqrt[4]{\frac{p}{q-p}}.$

4. Let the tangents at $P$ and $Q$ meet in $R$.

Then $RPA$ = angle in $PBA$ = angle in $QBA$ = $RQA$.

$\therefore RP = RQ.$ $\therefore R$ is on the radical axis, *i.e.* on $AB$.

5. Join $HT$ and produce it to $L$ so that $HL = AA'$. Then Bes. *El.* xi. $ML = MS$, and the angle $TML = TMS = HMT'$.

$\therefore$ the angle $HML = OMO'$, and the triangles $HML$, $OMO'$ are equal in all respects. $\therefore OO' = HL = AA'$.

6. Let $t_1$ be the time between projection and 1st impact, $t_2$ the time between the 1st and 2nd impacts.

Then $a = v\cos\theta \,.\, t_1.$ $\therefore t_1 = \frac{a}{v\cos\theta}.$

Hor. vel. just before 1st impact $= v\cos\theta$.

Vert. „ , „ „ $= v\sin\theta - gt_1.$

After impact the hor. vel. $= ev\cos\theta$,

$$\therefore a = ev\cos\theta \,.\, t_2, \therefore t_2 = \frac{a}{ev\cos\theta}.$$

Now the vert. vel. is not affected by the impact, and just before the 2nd impact it $= v\sin\theta - g(t_1 + t_2)$, and this must $= 0$ since the motion is horizontal,

$$\therefore v\sin\theta = \frac{ag}{v\cos\theta}\left(1 + \frac{1}{e}\right). \therefore \sin 2\theta = \frac{2ga}{v^2} \,.\, \frac{1+e}{e}.$$

7. Let $ABC$, $A'B'C'$ be the triangular faces, $G$ and $G'$ their C. of G Then by supposing the wedge to consist of triangular laminæ parallel to $ABC$ we see that its C. of G. will be at $H$ the middle point of $GG'$.

Now the C. of G. of 3 equal weights at $A$, $B$, $C$ is at $G$; and the C. of G. of 3 weights equal to these placed at $A'B'C'$ is at $G'$. $\therefore$ the C. of G. of the 6 equal weights is at $H$.

## Paper LXXV.

1. $x^3 - a^3 = xyz$. $\therefore x(x^2 - yz) = a^3$. $\therefore \frac{a^3}{x} = x^2 - yz$.

Similarly $\frac{b^3}{y} = y^2 - zx$, and $\frac{c^3}{z} = z^2 - xy$,

$$\therefore \frac{d^3}{x+y+z} = \frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} = x^2 + y^2 + z^2 - yz - zx - xy,$$

$$\therefore d^3 = x^3 + y^3 + z^3 - 3xyz.$$

$$= a^3 + b^3 + c^3.$$

2. The number $6n$ can be made up of

$0, 1, 6n-1$; $0, 2, 6n-2$; . . . . . $0, 3n-1, 3n+1$;
$1, 2, 6n-3$; . . . . . $1, 3n-1, 3n$;
$2, 3, 6n-5$; . . . . . $2, 3n-2, 3n$;
. . . . . .

$\therefore$ the number of ways is

$$(3n-1) + (3n-2) + (3n-4) + (3n-5) + (3n-7) + (3n-8) + \ldots$$
$$= 6n^2 - (1 + 4 + 7 + \ldots) - (2 + 5 + 8 + \ldots)$$
$$= 6n^2 - \frac{n}{2}\{2 + (n-1)3\} - \frac{n}{2}\{4 + (n-1)3\}$$
$$= 6n^2 - 3n^2 = 3n^2.$$

Now the whole number of ways in which 3 tickets can be drawn is

$$\frac{6n(6n-1)(6n-2)}{\lfloor 3}.$$

$$\therefore \text{the chance} = \frac{3n^2 \lfloor 3}{6n(6n-1)(6n-2)} = \frac{3n}{(6n-1)(6n-2)}.$$

3. $\frac{AP}{AB} = \frac{\sin ABP}{\sin APB} = \frac{\cos A}{\sin C}$. $\therefore AP = \frac{c}{\sin C} \cdot \cos A$.

Similarly $BP = \frac{a}{\sin A} \cdot \cos B$; $CP = \frac{b}{\sin B} \cdot \cos C$.

$\therefore$ the required condition is that any two of the quantities $\cos A$, $\cos B$, $\cos C$ must be together greater than the third.

$$\cos a = \frac{BP^2 + CP^2 - AP^2}{2BP \,.\, CP} = \frac{\cos^2 B + \cos^2 C - \cos^2 A}{2\cos B \cos C},$$

$$\therefore \frac{\cos a}{\cos A} = \frac{\cos^2 B + \cos^2 C - \cos^2 A}{2 \cos A \cos B \cos C}.$$

By symmetric changes we can write down the values of $\frac{\cos \beta}{\cos B}, \frac{\cos \gamma}{\cos C}$,

$$\therefore 1 + \frac{\cos a}{\cos A} + \frac{\cos \beta}{\cos B} + \frac{\cos \gamma}{\cos C} = 1 + \frac{\cos^2 A + \cos^2 B + \cos^2 C}{2 \cos A \cos B \cos C}$$

$$= \tfrac{1}{2} \sec A \sec B \sec C. \text{ Todh. } \textit{Trig.} \text{ Art. 115.}$$

4. In the question the lengths of the sides are supposed to be capable of varying. Suppose we take any polygon of 5 sides, $ABCDEF$, and let the middle points of all except $AF$ be fixed. First suppose $A$ to move for a given distance along a straight line. Then since the middle point of $AB$ is fixed, the point $B$ will move the same distance along a straight line through $B$ parallel to the former, but the direction in which $B$ moves will be opposite to that in which $A$ moves. Similarly the point $C$ will move through the same distance in a direction opposite to that of $B$, *i.e.* in the same direction as that in which $A$ moves. By proceeding in this manner we see that if we consider $A$ as the 1st angular point, then all the angular points move through the same distance as $A$, the odd moving in the same, and the even in the opposite direction to that in which $A$ moves. $\therefore$ the last point $F$ moves in the same direction as $A$. $\therefore$ the whole line $AF$, and $\therefore$ also its middle point $H$ moves in the same direction and through the same space.

The above demonstration is evidently true when the polygon has any odd number of sides, and when the point $H$ describes any curve instead of a straight line.

5. Project the ellipse into a circle, and denote corresponding points by small letters. Then evidently the polar of $e$ passes through $f$, and $ef$ bisects the angle $pfq'$.

$$\therefore \frac{ep^2}{eq^2} = \frac{fp^2}{fq^2} = \frac{fp \,.\, fq'}{fq \,.\, fp'}. \quad \therefore \left(\frac{EP}{EQ}\right)^2 = \frac{FP \,.\, FQ'}{FQ \,.\, FP'}.$$

6. Tripos 1878. Tuesday morning. No. 1.

7. Tripos 1875. Tuesday morning. No. 10.

## Paper LXXVI.

1. Let $S$ denote the stock which he transfers. Then he takes $S$ from £1,583 17*s.* 11*d.* and he adds $\frac{21}{98}S$ to £982 12*s.* 6*d.*; and since these sums each bring in the same income,

$$\therefore 3(£1{,}583\ 17s.\ 11d. - S) = \tfrac{7}{2}(£982\ 12s.\ 6d. + \tfrac{21}{98}S),$$

$$\therefore S = £210.$$

$\therefore$ the stock remaining in the 3 per cents. = £1,373 17*s.* 11*d.*

2. $$\frac{1}{2N^2 + x} = \frac{1}{2N^2} \cdot \frac{1}{1 + \frac{x}{2N^2}} = \frac{1}{2N^2}\left(1 + \frac{x}{2N^2}\right)^{-1}$$

$$= \frac{1}{2N^2}\left\{1 - \frac{x}{2N^2} + \frac{x^2}{4N^4} - \frac{x^3}{8N^6} + \dots\right\},$$

$$\therefore N + \frac{x}{4N} + \frac{Nx}{2(2N^2 + x)} = N + \frac{x}{2N} - \frac{x^2}{8N^3} + \frac{x^3}{16N^5} - \frac{x^4}{32N^7} + \dots$$

$$\sqrt{N^2 + x} = N\left(1 + \frac{x}{N^2}\right)^{\frac{1}{2}} = N + \frac{x}{2N} - \frac{x^2}{8N^3} + \frac{x^3}{16N^5} - \frac{5x^4}{128N^7} + \dots$$

$$\therefore \sqrt{N^2 + x} - \left\{N + \frac{x}{4N} + \frac{Nx}{2(2N^2 + x)}\right\} = -\frac{x^4}{128N^7} + \dots$$

$\therefore$ the error is of the order $\frac{x^4}{N^7}$.

$$\sqrt{101} = \sqrt{100 + 1} = \sqrt{10^2 + 1}.$$

Put $x = 1$, $N = 10$ in the above formula.

$$\therefore \sqrt{101} = 10 + \frac{1}{40} + \frac{10}{402} \text{ nearly}$$

$$= 10 + \frac{201 + 200}{8040} = 10\,\frac{401}{8040}.$$

Here we neglect $-\frac{1}{128 \cdot 10^7}$, which is $< \frac{1}{10^9}$,

$\therefore$ the result is correct to 8 places of decimals.

3. (1) $\cos(2A+B+C)+\cos(B+C) = 2\cos(A+B+C)\cos A,$

$\cos(A+2B+C)+\cos(C+A) = 2\cos(A+B+C)\cos B,$

$\cos(A+B+2C)+\cos(A+B) = 2\cos(A+B+C)\cos C,$

$\therefore$ the given expression on the left hand

$$= 2\cos(A+B+C)\left\{\cos(A+B+C)+\cos A+\cos B+\cos C\right\} - 1$$

$$= 4\cos(A+B+C)\left\{\cos\frac{B+C}{2}\cos\frac{2A+B+C}{2}+\cos\frac{B+C}{2}\cos\frac{B-C}{2}\right\} - 1$$

$$= 8\cos(A+B+C)\cos\frac{B+C}{2}\cos\frac{C+A}{2}\cos\frac{A+B}{2} - 1.$$

(2) $$\sin B+\sin C-\sin(B+C),$$

$$= 2\sin\frac{B+C}{2}\left\{\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right\}$$

$$= 4\sin\frac{B+C}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$\therefore$ the given expression on the left hand

$$= 64\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}\times P,$$

where $$P = \sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}$$

$$= \tfrac{1}{2}\sin\frac{A+B}{2}\left\{\cos\frac{A-B}{2}-\cos\frac{A+B+2C}{2}\right\}$$

$$= \tfrac{1}{4}\left\{\sin A+\sin B+\sin C-\sin(A+B+C)\right\}.$$

4. Let $O$, $O'$ be the centres of the equal circles. Suppose the radius of the circle, centre $A$, to be less than $AB$, and let this circle cut the circle, centre $O$ in $C$, and $O'$ in $D$. Join $BD$ and produce it to meet the circle, centre $O'$, in $E$. Join $AE$.

Then the angle $ADE$ = sup. of $ADB$ = sup. $ACB$ = $AEB$.

$\therefore$ $AD = AE$. $\therefore$ $E$ is a point on the circle, centre $A$.

5. Let $TR$ be the length of the chord, and let it cut $QQ'$ in $V$, the curve in $P$, and the directrix in $F$. Draw $QK$, $Q'K'$ at right angles to the directrix.

Then $TV \cdot VR = QV^2 = 4SP \cdot PV = 2SP \cdot TV$,

$$\therefore VR = 2SP = 2PF,$$

$$\therefore TR = TV + VR = 2PV + 2PF = 2VF = QK + Q'K' = SQ + SQ'.$$

6. Let $(h, k)$ be a point on the hyperbola $4xy = ab$. Then the equation to the tangent at $(h, k)$ is

$$2(xk + yh) = ab \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let this line cut the ellipse in the points $P$, $Q$, whose eccentric angles are $\theta$ and $\phi$. The equations to the normals at $P$ and $Q$ are

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2, \text{ and } \frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2.$$

If these intersect in $R$, the coordinates of $R$ are

$$\frac{a^2 - b^2}{a} \cdot \frac{\cos\frac{\theta + \phi}{2}}{\cos\frac{\theta - \phi}{2}} \cos\theta\cos\phi, \quad \frac{a^2 - b^2}{b} \cdot \frac{\sin\frac{\theta + \phi}{2}}{\cos\frac{\theta - \phi}{2}} \sin\theta\sin\phi,$$

$\therefore$ the equation to the diameter through $R$ is

$$by = ax \cdot \tan\frac{\theta + \phi}{2} \tan\theta\tan\phi.$$

Now since $P$ and $Q$ are on $2(xk + yh) = ab$,

$$\therefore k \cdot a\cos\theta + h \cdot b\sin\theta = \tfrac{1}{2}ab = ka\cos\phi + hb\sin\phi,$$

$$\therefore ak(\cos\phi - \cos\theta) = bh(\sin\theta - \sin\phi),$$

$$\therefore \frac{bh}{ak} = \frac{2\sin\frac{\theta - \phi}{2}\sin\frac{\theta + \phi}{2}}{2\sin\frac{\theta - \phi}{2}\cos\frac{\theta + \phi}{2}} = \tan\frac{\theta + \phi}{2}.$$

Again, by substituting the value of $y$ in terms of $x$ from (1) in the equation to the ellipse, we have

$$x^2\left(\frac{1}{a^2} + \frac{k^2}{b^2h^2}\right) - \frac{ak}{bh^2}x + \frac{a}{4h^2} = 1,$$

. . if $x_1$, $x_2$ be the roots of this equation,

$$a\cos\theta \cdot a\cos\phi = x_1x_2 = \frac{a^2 - 4h^2}{4} \cdot \frac{a^2b^2}{a^2k^2 + b^2h^2}.$$

So $$b\sin\theta \,.\, b\sin\phi = y_1 y_2 = \frac{b^2 - 4k^2}{4} \cdot \frac{a^2b^2}{a^2k^2 + b^2h^2},$$

$$\therefore \tan\theta\tan\phi = \frac{b^2 - 4k^2}{a^2 - 4h^2},$$

$$\therefore \tan\frac{\theta+\phi}{2}\tan\theta\tan\phi = \frac{b}{a} \cdot \frac{b^2h - 4hk\,.\,k}{a^2k - 4hk\,.\,h}, \text{ and } 4hk = ab,$$

$$= \frac{b}{a} \cdot \frac{b(bh - ak)}{a(ak - bh)} = -\frac{b^2}{a^2},$$

$\therefore$ the equation to $CR$ is $bx + ay = 0$, which is a fixed diameter of the ellipse.

7. Tripos 1878. Tuesday morning. No. 10.

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## Paper LXXVII.

1. $$f^2+g^2-h^2 = 2\{a_3^2 - a_3(a_1+a_2) + a_1a_2\} + 2\{b_3^2 - b_3(b_1+b_2) + b_1b_2\}$$

$$= 2(a_3 - a_1)(a_3 - a_2) + 2(b_3 - b_1)(b_3 - b_2),$$

$$\therefore 4f^2g^2 - (f^2 + g^2 - h^2)^2$$

$$= 4(a_2 - a_3)^2(b_3 - b_1)^2 + 4(a_3 - a_1)^2(b_2 - b_3)^2$$

$$- 8(a_3 - a_1)(a_2 - a_3)(b_3 - b_1)(b_2 - b_3)$$

$$= 4\{(a_2 - a_3)(b_3 - b_1) - (a_3 - a_1)(b_2 - b_3)\}^2,$$

from which we obtain the required result.

2. Let $O$ be the centre of the inscribed circle.

Then $$\frac{a}{OB} = \frac{\cos\frac{A}{2}}{\sin\frac{C}{2}}; \quad \frac{a}{OC} = \frac{\cos\frac{A}{2}}{\sin\frac{B}{2}};$$

$$\frac{OC}{OA} = \frac{\sin\frac{A}{2}}{\sin\frac{C}{2}}; \quad r = a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$$

$$\therefore l = \frac{1}{OA} = \frac{1}{OC} \cdot \frac{\sin\frac{A}{2}}{\sin\frac{C}{2}} = \frac{1}{a} \frac{\cos\frac{A}{2}\sin\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}},$$

So
$$m = \frac{1}{OB} = \frac{1}{a} \cdot \frac{\cos\frac{A}{2}}{\sin\frac{C}{2}} = \frac{1}{a} \cdot \frac{\cos\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}},$$

$$n = \frac{1}{OC} = \frac{1}{a} \frac{\cos\frac{A}{2}}{\sin\frac{B}{2}} = \frac{1}{a} \frac{\cos\frac{A}{2}\sin\frac{C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}},$$

$$\therefore \rho^3 - \rho(l^2 + m^2 + n^2) - 2lmn$$

$$= \frac{1}{a^3} \cdot \frac{\cos^3\frac{A}{2}}{\sin^3\frac{B}{2}\sin^3\frac{C}{2}} \left\{1 - \left(\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2}\right) - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right\}$$

$$= \ldots\ldots \left\{\cos^2\frac{A}{2} - \sin^2\frac{B}{2} - \sin\frac{C}{2}\left(\sin\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\right)\right\}$$

$$= \ldots \left\{\cos\frac{A+B}{2}\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\left(\cos\frac{A+B}{2} + \cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right)\right\}$$

$$= 0.$$

3. Let $E, F, G, H, K, L$ be the middle points of $BC, CD, DB, BA, AC, AD$.

Then $$EA^2 + ED^2 = 2EL^2 + 2DL^2,$$

$$\therefore 4EL^2 = 2EA^2 + 2ED^2 - 4DL^2$$
$$= AB^2 + AC^2 - 2EC^2 + DB^2 + DC^2 - 2EC^2 - AD^2$$
$$= AB^2 + AC^2 + BD^2 + DC^2 - AD^2 - BC^2.$$

So $4HF^2 = BD^2 + AC^2 + BC^2 + AD^2 - AB^2 - CD^2$

and $4GK^2 = AB^2 + CD^2 + BC^2 + AD^2 - BD^2 - AC^2.$

$\therefore$ by addition we obtain the result required.

4. From $S$ draw $SY$ perpendicular to the tangent at $P$, and produce $SY$ to meet the directrix in $D$. Let $DQ$, $DQ'$ be the tangents from $D$. Then since $D$ is on the directrix, $QQ'$ passes through the focus, and is perpendicular to $SD$. $\therefore$ the tangent at $P$ is parallel to $QQ'$. $\therefore$ $DP$ passes through the centre.

5. Tripos 1875. Wednesday morning. No. 12.

6. Let $P$ be the point, and $PT$ the direction of projection, $PSP'$ a focal chord and $PN$, $PK$ horizontal and vertical lines through $P$. Let $TPN = \alpha$, $P'PN = \beta$, and let $v$ be the velocity of projection, and $t$ the time of flight from $P$ to $P'$, $P'$ being further from the directrix than $P$.

Then
$$PP' = \frac{2v^2}{g} \cdot \frac{\sin(\alpha - \beta)\cos\alpha}{\cos^2\beta},$$

and
$$\alpha - \beta = P'PT = TPK = \frac{\pi}{2} - \alpha,$$

$$\therefore t = \frac{2v}{g} \cdot \frac{\sin(\alpha - \beta)}{\cos\beta}$$

$$= \sqrt{\frac{2PP'\sin(\alpha - \beta)}{g\cos\alpha}} = \sqrt{\frac{2PP'}{g}}$$

$$= \text{time taken to fall from rest through } PP'.$$

7. 1st Geometrical Method.

Let $A$ be the vertex of the triangle, $S$ the fixed point in the base $BC$. Draw $SD$ and $SE$ at right angles to $AB$ and $AC$. Then $DE$ is a fixed straight line. Let any circle through $A$ and $S$ cut $AB$ in $L$ and $N$. Let $LN$ cut $ED$ in $M$, and join $SM$, $SA$.

Then since circles will go round $ALSN$, and $ADSE$,

$$\therefore \text{the angle } SNM = SAL = SEM.$$

$\therefore$ a circle will go round $SMEN$. $\therefore$ the angle $SMN = SEN =$ a right angle. Now $S$ is a fixed point, and the point $M$ always lies on the fixed line $DE$, and $SMN$ is a right angle, $\therefore$ $LN$ envelopes a parabola of which $S$ is the focus and $DE$ the tangent at the vertex.

2nd Analytical Method.

Let the equation of $BC$ be $\quad ax + by = 1$

and the equation of $LN$ be $\quad ax + \beta y = 1$

" " of $AS$ be $\quad \frac{x}{l} = \frac{y}{m} = r.$

Then the equation to the circle round $ALSN$ is

$$x^2 + y^2 + 2xy\cos w - \frac{x}{a} - \frac{y}{\beta} = 0,$$

$$\therefore r^2 - r\left(\frac{l}{a} + \frac{m}{\beta}\right) = 0.$$

$$\therefore r = \frac{l}{a} + \frac{m}{\beta} = \left(\frac{l}{a} + \frac{m}{\beta}\right)(ax + \beta y).$$

$\therefore$ the equation to $LN$ is

$$(ax + \beta y)(l\beta + m\alpha) - ra\beta = 0,$$

or $$a^2mx + \beta^2 ly + a\beta(lx + my - r) = 0.$$

$\therefore$ the equation to the envelope is

$$(lx + my - r)^2 = 4lmxy,$$

or $$\sqrt{lx} + \sqrt{my} = \sqrt{r},$$

which is the equation to a parabola touching the axes, the distances of the points of contact from the origin being $\frac{r}{l}$, $\frac{r}{m}$.

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## Paper LXXVIII.

1. $$220 = 2 \cdot 2 \cdot 5 \cdot 11.$$

$\therefore$ the aliquot parts of 220 are

1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110, and their sum = 284.

$$284 = 2 \cdot 2 \cdot 71.$$

$\therefore$ the aliquot parts of 284 are

1, 2, 4, 71, 142, and their sum = 220.

2. Put each of the given expressions $= \lambda$.

Then $ax + ly = \lambda$ (1); $cx + dy = \lambda^3$ (2); $ex + fy = \lambda^5$ (3).

From (1) and (2) $x(ad - bc) = \lambda(d - b\lambda^2)$, $y(ad - bc) = \lambda(a\lambda^2 - c)$

$$\therefore \text{if for } \lambda \text{ we write } \sqrt{k},\quad x = \frac{d - bk}{ad - bc}\sqrt{k}.$$

To find $k$, we must substitute the values of $x$ and $y$ in (3),

$\therefore (ad - bc)\lambda^5 = ex(ad - bc) + fy(ad - bc)$

$= e\lambda(d - b\lambda^2) + f\lambda(a\lambda^2 - c)$. $\therefore$ either $\lambda = 0$, $\therefore x = 0$, $y = 0$,

or $(ad - bc)\lambda^4 - (fa - be)\lambda^2 + fc - de = 0$,

or $(ad - bc)k^2 - (fa - be)k + fc - de = 0$. . . . $(A)$

Let $a = 7$, $b = -11$, $c = 1$, $d = 1$, $e = 1$, $f = 9$.

Substituting these values in $(A)$ we have

$$18k^2 - 74k + 8 = 0, \quad \therefore k = 4 \text{ or } \tfrac{1}{9},$$

$$\therefore x = 5 \text{ or } \tfrac{10}{243}; \quad y = 3 \text{ or } -\tfrac{1}{243}.$$

Again, $x - y = \lambda$ (1); $4x - 5y = \lambda^{n+1}$ (2); $3x - 2y = \lambda^{2n+1}$ (3).

From (1) and (2) $x = \lambda(5 - \lambda^n)$, $y = \lambda(4 - \lambda^n)$ . . . $(B)$

From (3) $\lambda^{2n+1} = 3\lambda(5 - \lambda^n) - 2\lambda(4 - \lambda^n)$,

$\therefore$ either $\lambda = 0$, $\therefore x = 0$, $y = 0$,

or $$\lambda^{2n} + \lambda^n - 7 = 0, \quad \therefore \lambda^n = \frac{-1 \pm \sqrt{29}}{2},$$

from $(B)$ we obtain the values of $x$ and $y$.

3. Let $A$, $B$, $C$ be the centres of the three circles which touch externally, $r_1$, $r_2$, $r_3$ their radii, $P$ the centre of the circle whose radius is $R$.

Then $\triangle ABC = BPC + CPA + APB$ . . . $(A)$

$\therefore$ using the formula $S = \sqrt{s(s-a)(s-b)(s-c)}$,

$\triangle ABC = \sqrt{r_1r_2r_3(r_1 + r_2 + r_3)}$, $\triangle BPC = \sqrt{r_2r_3R(r_2 + r_3 + R)}$,

$\triangle CPA = \sqrt{r_3r_1R(r_3 + r_1 + R)}$, $\triangle BPA = \sqrt{r_1r_2R(r_1 + r_2 + R)}$,

$\therefore$ substituting in $(A)$ and dividing both sides by $r_1r_2r_3R$, we obtain the result (1). See also LXVIII., 3.

Now let the angle $BPC = \alpha$, $CPA = \beta$, $APB = \gamma$, $\therefore \alpha + \beta + \gamma = 2\pi$.

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma = 1$. . . $(B)$

Now $\cos\alpha = \dfrac{(r_2+R)^2+(r_3+R)^2-(r_2+r_3)^2}{2(r_2+R)(r_3+R)} = \dfrac{R^2+R(r_2+r_3)-r_2r_3}{(r_2+R)(r_3+R)}$

$$= \frac{(R+r_2)(R+r_3) - 2r_2r_3}{(r_2+R)(r_3+R)} = 1 - \frac{2r_2r_3}{(r_2+R)(r_3+R)}$$

$$= 1 - 2\lambda . \frac{R+r_1}{r_1}, \text{ where } \lambda = \frac{r_1r_2r_3}{(r_1+R)(r_2+R)(r_3+R)},$$

and similarly we can obtain

$$\cos\beta = 1 - 2\lambda \frac{R+r_2}{r_2}, \cos\gamma = 1 - 2\lambda \frac{R+r_3}{r_3}$$

Substitute these values in $(B)$, and denote by $\Sigma(r)$ the sum of the expressions obtained by giving to $r$ in succession the values $r_1$, $r_2$, $r_3$.

$$\therefore 1 = 3 - 4\lambda\Sigma\left(\frac{R+r}{r}\right) + 4\lambda^2\Sigma\left(\frac{R+r}{r}\right)^2$$

$$- 2\left\{1 - 2\lambda\Sigma\left(\frac{R+r}{r}\right) + 4\lambda^2\Sigma\frac{(R+r_1)(R+r_2)}{r_1r_2} - 8\lambda^2\frac{(R+r_1)(R+r_2)(R+r_3)}{r_1r_2r_3}\right\},$$

$$\therefore 0 = 4\lambda^2\Sigma\left(\frac{R+r}{r}\right)^2 - 8\lambda^2\Sigma\frac{(R+r_1)(R+r_2)}{r_1r_2} + 16\lambda^2,$$

$$\therefore \Sigma\left(1+\frac{R}{r}\right)^2 - 2\Sigma\left\{1 + R\left(\frac{1}{r_1}+\frac{1}{r_2}\right) + \frac{R^2}{r_1r_2}\right\} + 4 = 0,$$

$$\therefore 3 + 2R\Sigma\left(\frac{1}{r}\right) + R^2\Sigma\left(\frac{1}{r^2}\right) - 2\left\{3 + 2R\Sigma\left(\frac{1}{r}\right) + R^2\Sigma\left(\frac{1}{r_1r_2}\right)\right\} + 4 = 0,$$

$$\therefore \frac{1}{R^2} - \frac{2}{R}\Sigma\left(\frac{1}{r}\right) + \Sigma\left(\frac{1}{r^2} - \frac{1}{r_1r_2}\right) = 0,$$

$$\therefore \frac{1}{R} = \Sigma\left(\frac{1}{r}\right) \pm 2\sqrt{\Sigma\left(\frac{1}{r_1r_2}\right)}$$

$$= \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm 2\sqrt{\frac{1}{r_2r_3} + \frac{1}{r_3r_1} + \frac{1}{r_1r_2}}.$$

Again, the area of the triangle $BPC$ can be written in either of the forms

$$\tfrac{1}{2}d(r_2+r_3), \text{ or } \sqrt{r_2r_3R(r_2+r_3+R)},$$

$$\therefore \tfrac{1}{2}d(r_2+r_3) = \sqrt{r_2r_3R(r_2+r_3+R)},$$

Similarly $\frac{1}{2}d'(r_2 + r_3) = \sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}$,

$$\therefore \frac{1}{2}\left(\frac{d}{R} - \frac{d'}{r_1}\right) = \frac{\sqrt{r_2 r_3}}{r_2 + r_3}\left\{\sqrt{\frac{r_2 + r_3 + R}{R}} - \sqrt{\frac{r_1 + r_2 + r_3}{r_1}}\right\}.$$

Now the expression on the right is equal to 1 if

$$\sqrt{1 + \frac{r_2 + r_3}{R}} = \frac{r_2 + r_3}{\sqrt{r_2 r_3}} + \sqrt{1 + \frac{r_2 + r_3}{r_1}},$$

*i.e.* if $1 + \frac{r_2 + r_3}{R} = \frac{(r_2 + r_3)^2}{r_2 r_3} + 2\frac{r_2 + r_3}{\sqrt{r_2 r_3}} \cdot \frac{\sqrt{r_1 + r_2 + r_3}}{\sqrt{r_1}} + 1 + \frac{r_2 + r_3}{r_1}$,

*i.e.* if $\frac{r_2 + r_3}{R} = \frac{(r_2 + r_3)^2}{r_2 r_3} + 2(r_2 + r_3)\sqrt{\frac{1}{r_2 r_3} + \frac{1}{r_3 r_1} + \frac{1}{r_1 r_2}} + \frac{r_2 + r_3}{r_1}$,

*i.e.* if $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + 2\sqrt{\frac{1}{r_2 r_3} + \frac{1}{r_3 r_1} + \frac{1}{r_1 r_2}}$,

which we know to be true,

$$\therefore \frac{d}{R} - \frac{d'}{r_1} = 2.$$

4. Let $R$, $O$ be the radius and centre of the circum-circle,

$r_a$, $O_a$ ,, ,, escribed circle.

Then $OO_a^2 = R^2 + 2Rr_a$. Now $O_aP.O_aQ$ = square of tangent from $O_a$ to the circum-circle.

$\therefore O_aP.O_aQ = OO_a^2 - R^2 = 2Rr_a$; and $O_aP = r_a \therefore O_aQ = 2R$.

†5. Let $O$ be the orthocentre, $D$, $E$, $F$ the feet of the perpendiculars on the sides from $A$, $B$, $C$. On $BC$ take $BP = CD$. On $CA$ take $AQ = CE$. On $AB$ take $AR = BF$. Then $PQR$ is the triangle required. For it is easily proved that the perpendiculars drawn from any point in the diagonal of a parallelogram to two adjacent sides are inversely proportional to the sides to which they are drawn, and conversely, any point from which perpendiculars drawn to two adjacent sides of a parallelogram are inversely proportional to the sides must lie on the diagonal through the intersection of these sides.

Let $S$ be the circum-centre, $A'$, $B'$, $C'$ the middle points.

† This solution is by the Rev. G. Richardson, Winchester.

Then $\frac{AR}{AQ} = \frac{BF}{CE} = \frac{BO}{CO}$ from similar triangles $BFO$, $CEO$,

$$= \frac{SB'}{SC'} \quad \text{,,} \quad \text{,,} \quad BOC, B'SC'.$$

$\therefore$ $S$ must lie on the diagonal of the parallelogram of which $AR$, $AQ$ are adjacent sides, and $\therefore$ $SA$ bisects the other diagonal $QR$. Similarly for $SB$, $SC$.

6. Join $PQ$ meeting the axis in $N$, and draw the ordinates $QC$, $NE'$. Then $AM \,.\, AC = AN^2$. Bes. *Parab.* Prop. xxi.

But $PM^2 = 4AS \,.\, AM$, $QC^2 = 4AS \,.\, AC$, $R'N^2 = 4AS \,.\, AN$,

$$\therefore\ PM^2 \,.\, QC^2 = R'N^4. \quad \therefore\ R'N^2 = PM \,.\, MV = ME^2.$$

$$\therefore\ R' \text{ coincides with } R.$$

7. Tripos 1875. Wednesday morning. No. 13.

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## Paper LXXIX.

1. Let $N$ denote the number of pounds which $B$ must pay per month. Then the total value of $B$'s payments to $A$ at the end of 12 months at $3\frac{1}{3}$ per cent

$$= 12N + \frac{3\frac{1}{3}}{100} N \frac{1 + 2 + \ldots + 12}{12} = 12\tfrac{13}{60} N.$$

Now the total value at the end of 12 months of the money which $A$ has paid to his landlord

$$= \text{£}\left\{40 + \frac{3\frac{1}{3}}{100} \cdot 10 \cdot \left(\frac{1 + 2 + 3}{4}\right)\right\} = \text{£}40\ 10s.$$

and one-tenth of this = £4 1$s$.

$$\therefore\ 12\tfrac{13}{60} N = \text{£}44\ 11s, \quad \therefore\ N = \text{£}3\ 12s.\ 11\tfrac{14}{73}\tfrac{5}{3}d.$$

2. $a + b + c + d = 0$, $\therefore\ (a + b)^3 = -(c + d)^3$,

$$\therefore\ 0 = (a + b)^3 + (c + d)^3$$
$$= a^3 + b^3 + c^3 + d^3 + 3\{ab(a + b) + cd(c + d)\}$$
$$= a^3 + b^3 + c^3 + d^3 - 3\{ab(c + d) + cd(a + b)\},$$

$$\therefore\ (a^3 + b^3 + c^3 + d^3)^2 = 9\{bcd + cda + dab + abc\}^2.$$

Again, $0 = (a + b + c + d)^2$

$$= a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)$$

$$= a^2 + b^2 + c^2 + d^2 + 2\{ab + cd + (a + b)(c + d)\},$$

$$\therefore\ (bc - ad)(ca - bd)(ab - cd)$$

$$= a^2b^2(c^2 + d^2) + c^2d^2(a^2 + b^2) - abcd(a^2 + b^2 + c^2 + d^2)$$

$$= a^2b^2(c^2 + d^2) + c^2d^2(a^2 + b^2) + abcd \,.\, 2\{ab + cd + (a + b)(c + d)\}$$

$$= a^2b^2(c + d)^2 + c^2d^2(a + b)^2 + 2abcd(a + b)(c + d)$$

$$= \{ab(c + d) + cd(a + b)\}^2.$$

3. Tripos 1878. Thursday afternoon. No. 2.

4. Tripos 1875. Wednesday morning. No 15.

5. Tripos 1875. 2nd Monday morning. No. 2.

6. Let $CR = m.CP$. Then the coordinates of $Q$ are $ma\cos\phi$, $mb\sin\phi$. Let $\xi, \eta$ be the coordinates of the point of contact of one of the tangents from $Q$. The equation to this tangent is $\frac{x\xi}{a^2} + \frac{y\eta}{b^2} = 1$. Since it passes through $Q$, we have

$$1 = \frac{\xi ma\cos\phi}{a^2} + \frac{\eta mb\sin\phi}{b^2}$$

$$= \frac{\xi}{a} ma\cos\phi + \frac{\eta}{b} m\sin\phi,$$

$$\therefore\ \frac{\xi}{a} = \frac{1}{m\cos\phi}\left(1 - \frac{\eta}{b} m\sin\phi\right).$$

Since $(\xi, \eta)$ is on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$$\therefore\ 1 = \frac{\eta^2}{b^2} + \frac{1}{m^2\cos^2\phi}\left(1 - \frac{2\eta}{b} m\sin\phi + \frac{\eta^2}{b^2} m^2\sin^2\phi\right)$$

$$\therefore\ \eta^2m^2 - 2\eta mb\sin\phi + b^2(1 - m^2\cos^2\phi) = 0,$$

$$\therefore \eta = b\left\{\frac{\sin\phi}{m} \pm \frac{\sqrt{m^2\cos^2\phi - \cos^2\phi}}{m}\right\}$$

$$= b\left\{\sin\phi \cdot \frac{1}{m} \pm \cos\phi \sqrt{1 - \frac{1}{m^2}}\right\}$$

$$= b\left\{\sin\phi \cos\left(\cos^{-1}\frac{CP}{CQ}\right) \pm \cos\phi \sin\left(\cos^{-1}\frac{CP}{CQ}\right)\right\}$$

$$= b \sin\left\{\phi \pm \cos^{-1}\frac{CP}{CQ}\right\}.$$

Similarly it may be shewn that

$$\xi = a \cos\left\{\phi \pm \cos^{-1}\frac{CP}{CQ}\right\}.$$

7. Let $w$ denote the weight of each pulley. Since there is equilibrium,

$$P = \frac{W}{2^n} + w\left(1 - \frac{1}{2^n}\right), \quad \therefore w = \frac{2^n P - W}{2^n - 1} \quad . \quad . \quad . \quad (1)$$

Let $f, f', f_1, f_2, \ldots f_n$ denote the accelerations of $P$, $W'$, and the pulleys beginning with the highest. Then by considering the spaces moved over in the same time, we have

$$f_n = f', f_{n-1} = 2.f' \quad . \quad . \quad . \quad f_1 = 2^{n-1}f', f = 2^n f'.$$

$\therefore$ the kinetic energy of the system at the end of time $t$

$$= \tfrac{1}{2}(2^n f'.t)^2.P' + \tfrac{1}{2}(2^{n-1}f'.t)^2.w + \ldots + \tfrac{1}{2}(f't)^2.w + \tfrac{1}{2}(f't)^2 W'.$$

$$= \tfrac{1}{2}f'^2t^2\left\{2^{2n}P' + W' + w\frac{2^{2n}-1}{2^2-1}\right\}$$

$$= \tfrac{1}{2}f'^2t^2\left\{2^{2n}P' + W' + \tfrac{1}{3}(2^n+1)(2^nP - W)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Now the work done by gravity upon the system in the time $t$

$$= \tfrac{1}{2}gt^2.gP' - \tfrac{1}{2}gt^2(f_1w + f_2w + \ldots + f_nw + f'W')$$

$$= \tfrac{1}{2}f't^2g(2^nP' - W' - 2^nP + W) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

$\therefore$ equating (2) and (3), and remembering that $f = 2^n f'$, we obtain the result required.

## Paper LXXX.

1. Put each of the given expressions $= \lambda$.

Then $\lambda = \dfrac{Pa^2 + 2Qab + Rb^2}{D_1}$

where $D_1 = p(a^4 + 2a^2bc + b^2c^2) + 2q(a^3b + ab^2c - a^3b - ab^2c) + r(a^2b^2 - 2a^2b^2 + a^2b^2)$

$= p(a^2 + bc)^2.$

Again $\lambda = \dfrac{Pac + Q(bc - a^2) - Rab}{D_2}$

where $D_2 = p(a^3c + abc^2 - a^3c - abc^2) + q\{2a^2bc + (bc - a^2)^2 + 2a^2bc\} + r(ab^2c - ab^2c + a^3b - a^3b)$

$= q(a^2 + bc)^2.$

Also $\lambda = \dfrac{Pc^2 - 2Qca + Ra^2}{D_3}$

where $D_3 = p(a^2c^2 - 2a^2c^2 + a^2c^2) + 2q(abc^2 - abc^2 + a^3c - a^3c) + r(b^2c^2 + 2a^2bc + a^4)$

$= r(a^2 + bc)^2.$

$\therefore \dfrac{p}{Pa^2 + 2Qab + Rb^2} = \dfrac{q}{Pac + Q(bc - a^2) - Rab}$

$= \dfrac{r}{Pc^2 - 2Qca + Ra^2}.$

2. The number of ways is evidently the number of combinations of $4m$ things taken $2m$ at a time, when there are $m$ alike of each of 4 sets. $\therefore$ Todh. *Alg.* Art 811, the required number is the coefficient of $x^{2m}$ in the expansion of

$$\frac{1 - x^{m+1}}{1 - x} \cdot \frac{1 - x^{m+1}}{1 - x} \cdot \frac{1 - x^{m+1}}{1 - x} \cdot \frac{1 - x^{m+1}}{1 - x},$$

$$= \text{coef. of } x^{2m} \text{ in } \left(\frac{1 - x^{m+1}}{1 - x}\right)^4,$$

$$= \text{coef. of } x^{2m} \text{ in } \left\{1 - 4x^{m+1} + \ldots\ldots\right\}\left\{1 + \ldots\ldots\right.$$

$$\left. + \frac{m(m+1)(m+2)}{\lfloor 3} x^{m-1} + \frac{(2m+1)(2m+2)(2m+3)}{\lfloor 3} x^{2m} + \ldots\right\}$$

$$= \frac{(2m+1)(2m+2)(2m+3)}{\lfloor 3} - 4 \cdot \frac{m(m+1)(m+2)}{\lfloor 3}$$

$$= \frac{m+1}{3}\left\{(2m+1)(2m+3) - 2m(m+2)\right\}$$

$$= \tfrac{1}{3}(m+1)(2m^2+4m+3).$$

3. Consider the expression

$$a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2).$$

If we equate it to zero, it is satisfied by $a = b$. $\therefore a - b$ is a factor. Similarly $b - c$, $c - a$ are factors.

$$\therefore \text{the expression} \equiv (b - c)(c - a)(a - b)\left\{A(a^2 + b^2 + c^2)\right.$$
$$\left. + B(bc + ca + ab)\right\}.$$

Equating coefficients of corresponding terms on both sides we find $A = 0$, $B = -1$,

$$\therefore \frac{a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2)}{(a - b)(b - c)(c - a)} \equiv -(bc + ca + ab) \quad (C).$$

Since this is true whatever be the values of $a$, $b$, $c$,

$$\text{let } a = e^{\alpha i},\ b = e^{\beta i},\ c = e^{\gamma i},$$

$$\therefore a - b = \cos\alpha - \cos\beta + i(\sin\alpha - \sin\beta)$$

$$= 2\sin\frac{\beta - \alpha}{2}\sin\frac{\alpha + \beta}{2} + 2i\sin\frac{\alpha - \beta}{2}\cos\frac{\alpha + \beta}{2}$$

$$= 2i\sin\frac{\alpha - \beta}{2}\left(\cos\frac{\alpha + \beta}{2} + i\sin\frac{\alpha + \beta}{2}\right)$$

$$= 2i\sin\frac{\alpha - \beta}{2} \cdot e^{\frac{\alpha+\beta}{2} \cdot i}.$$

So $a^2 - b^2 = 2i\sin(\alpha - \beta) \cdot e^{(\alpha+\beta)i}$,

$$\therefore \frac{a^3(b^2-c^2)}{(a-b)(b-c)(c-a)} = \frac{2i\sin(\beta-\gamma)\,.\,e^{(3\alpha+\beta+\gamma)i}}{-8i\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\,.\,e^{(\alpha+\beta+\gamma)i}}$$

$$= -\tfrac{1}{4}\frac{\sin(\beta-\gamma)e^{2\alpha i}}{\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}}\,.$$

$\therefore$ from $(C)$ we have

$$\frac{\Sigma\{\sin(\beta-\gamma)e^{2\alpha i}\}}{4\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}} = \Sigma\{e^{(\beta+\gamma)i}\},$$

$\therefore$ putting $e^{2\alpha i} = \cos 2\alpha + i\sin 2\alpha$, &c., and equating real parts, we have

$$2\{\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)\}$$

$$= \Sigma\frac{\cos 2\alpha\sin(\beta-\gamma)}{2\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}} = \Sigma\frac{\cos 2\alpha\cos\frac{1}{2}(\gamma-\beta)}{\sin\frac{\alpha-\beta}{2}\sin\frac{\gamma-\alpha}{2}}$$

$$= \Sigma\frac{\cos 2\alpha\cos\left\{\frac{1}{2}(\alpha-\beta)+\frac{1}{2}(\gamma-\alpha)\right\}}{\sin\frac{\alpha-\beta}{2}\sin\frac{\gamma-\alpha}{2}}$$

$$= \Sigma\cos 2\alpha\left\{\cot\tfrac{1}{2}(\gamma-\alpha)\cot\tfrac{1}{2}(\alpha-\beta)-1\right\}.$$

By transposition, we obtain the required result.

The problem might be solved more directly as follows.

Let $x = \cos 2\alpha\sin(\beta-\gamma)+\cos 2\beta\sin(\gamma-\alpha)+\cos 2\gamma\sin(\alpha-\beta)$,

and $y = \sin(\beta-\gamma)+\sin(\gamma-\alpha)+\sin(\alpha-\beta)$

$$= -4\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta).$$

Todh. *Trig.* viii. Ex. 3.

$$\therefore x - y\cos 2\gamma = \sin(\beta-\gamma)(\cos 2\alpha-\cos 2\gamma)+\sin(\gamma-\alpha)(\cos 2\beta-\cos 2\gamma)$$

$$= -2\sin(\beta-\gamma)\sin(\alpha-\gamma)\sin(\alpha+\gamma)$$
$$-2\sin(\gamma-\alpha)\sin(\gamma-\beta)\sin(\beta+\gamma)$$

$$= 2\sin(\beta-\gamma)\sin(\gamma-\alpha)\{\sin(\alpha+\gamma)-\sin(\beta+\gamma)\}$$

$$= 4\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\alpha+\beta+2\gamma)$$

$$= -4y\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha)\cos\left(\frac{\alpha+\beta}{2}+\gamma\right)$$

$$= -2y\left\{\cos\tfrac{1}{2}(\beta-\alpha)+\cos\left(\frac{\alpha+\beta}{2}-\gamma\right)\right\}\cos\left(\frac{\alpha+\beta}{2}+\gamma\right)$$

$$= -y\{\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)+\cos 2\gamma\}$$

$$\therefore\ x = y\{\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)\}.$$

From this the required result follows as in the previous method.

4. Let $A'B'C'D'$ be the given quadrilateral, and let $A$, $B$, $C$, $D$ be the points of contact of the circle with $A'B'$, $A'D'$, $C'D'$, $B'C'$. Let $AC$ and $BD$ intersect in $O$. Then the angle $BD'C = D'BC$, and $ABD = B'AD$,

$$\therefore\ BAO + ABO = \tfrac{1}{2}(\pi - D') + \tfrac{1}{2}(\pi - B') = \pi - \tfrac{1}{2}(B' + D')$$

$$= \pi - \frac{\pi}{2}, \text{ since a circle can be described about } A'B'C'D'$$

$\therefore$ $AOB$ is a right angle. $\therefore$ $AC$ and $BD$ intersect at right angles. Now let $K$, $L$, $M$, $N$ be the middle points of $AB$, $BC$, $CD$, $DA$. Then $KL$ and $MN$ are each of them parallel to $AC$, and $KN$, $LM$ are each parallel to $BD$, $\therefore$ $KLMN$ is a rectangle, and a circle can be described round $K$, $L$, $M$, $N$.

Let $O$, $I$ be the centres of the circum- and inscribed circles of $A'B'C'D'$. Join $A'I$ meeting the circum-circle in $E$. Then since $A'I$ bisects the angle $B'A'D'$, $\therefore$ $E$ is the middle point of the arc $B'D'$. Produce $EO$ to meet the circle in $F$.

Then $r = A'I\sin\dfrac{A'}{2}$, $r = C'I\sin\dfrac{C'}{2} = C'I\cos\dfrac{A'}{2}$, for $A' + C' = \pi$,

$$\therefore\ A'I\,.\,IC' = 2r^2\operatorname{cosec} A'.$$

Since $F$ is the middle point of the arc $B'D'$, $\therefore$ $C'F$ bisects the angle $B'C'D'$ and passes through $I$.

$$\therefore\ A'I\,.\,IE = C'I\,.\,IF = R^2 - \delta^2.$$

Also $A'C' = 2R\sin A'B'C' = 2R\sin A'EC' = 2R\dfrac{C'I}{EI}$

and $$B'D' = 2R\sin B'A'D' = \frac{4Rr^2}{A'I\,.\,IC'}$$

$$\therefore\ A'C'\,.\,B'D' = \frac{8R^2\,.\,r^2}{A'I\,.\,IE} = \frac{8R^2r^2}{R^2-\delta^2}.$$

Again, let $A'B' = a$, $B'C' = b$, $C'D' = c$, $D'A' = d$. Let $A'C'$, $B'D'$ intersect in $P$. Then the area of $A'B'C'D'$

$$= \sqrt{abcd}. \quad \therefore \sin B' = \frac{2\sqrt{abcd}}{ab + cd}.$$

Now $A'P = \dfrac{a \sin A'B'D'}{\sin \theta} = \dfrac{ad \sin B'}{A'C' \sin \theta}$

So $C'P = \dfrac{bc \sin B'}{A'C' \sin \theta}$,

$$\therefore \frac{(ad + bc) \sin B'}{\sin \theta} = A'C'(A'P + PC') = A'C'^2$$

$$= \frac{(ac + bd)(ad + bc)}{ab + cd},$$

$$\therefore \sin \theta = \frac{ab + cd}{ac + bd} \sin B = \frac{2\sqrt{abcd}}{ac + bd}$$

$$\therefore \cos^2 \theta = \frac{(ac + bd)^2 - 4abcd}{(ac + bd)^2} = \left(\frac{ac \sim bd}{ac + bd}\right)^2.$$

If $ac > bd$, $\tan^2 \dfrac{\theta}{2} = \dfrac{1 - \cos \theta}{1 + \cos \theta} = \dfrac{bd}{ac} = \dfrac{abcd}{(ac)^2} = \dfrac{S^2}{(ac)^2}$.

If $ac < bd$ $\tan^2 \dfrac{\theta}{2} = \dfrac{ac}{bd} = \dfrac{abcd}{(bd)^2} = \dfrac{S^2}{(bd)^2}$.

If $\theta = 90°$, $\dfrac{\theta}{2} = 45°$,

$$\therefore 1 = \tan \frac{\theta}{2} = \frac{S}{ac} = \frac{\sqrt{abcd}}{ac} = \sqrt{\frac{bd}{ac}} \therefore bd = ac.$$

5. Let $QG$ meet $Q'G'$ in $F$, and let $CD$ be conjugate to $CP$.

Then $QP \cdot PQ' = CD^2 = PG \cdot PG'$. $\therefore QP : PG :: PG' : PQ$,

$\therefore$ the triangles $PQG$, $PG'Q'$ are similar. $\therefore$ the angle $PQF = PG'F$.

$\therefore$ a circle will go round $PQG'F$.

$\therefore$ the angle $G'FQ = G'PQ =$ a right angle.

6. Since the parabola passes the origin, its equation is of the form

$$(\alpha x + \beta y)^2 + 2gx + 2fy = 0.$$

The equation to the directrix, Salm. *Con.* p. 269, is

$$x\beta(g\beta - f\alpha) - y\alpha(g\beta - f\alpha) = \tfrac{1}{2}(f^2 + g^2) \quad . \quad . \qquad (A)$$

If $y = 0$, $x = 0$ or $-\frac{2g}{a^2}$, $\therefore -\frac{g}{a^2} = h$.

If $x = 0$, $y = 0$ or $-\frac{2f}{\beta^2}$, $\therefore -\frac{f}{\beta^2} = k$.

$\therefore g\beta - f\alpha = \alpha\beta(\beta k - \alpha h)$; $f^2 + g^2 = \alpha^4 h^2 + \beta^4 k^2$,

$\therefore$ $(A)$ becomes $2\alpha\beta(\beta k - \alpha h)(\beta x - \alpha y) = \alpha^4 h^2 + \beta^4 k^2$ . $(B)$.

The equation to a straight line through the origin perpendicular to this is

$$\alpha x + \beta y = 0, \quad \therefore \frac{-\alpha}{y} = \frac{\beta}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (C).$$

$\therefore$ eliminating $\alpha$ and $\beta$ between $(B)$ and $(C)$ we have for the locus of the intersection

$$2xy(kx + hy)(x^2 + y^2) + h^2y^4 + k^2x^4 = 0.$$

By Salm. *Con.* p. 196, the equation to the axis is

$$\alpha x + \beta y = -\frac{\alpha g + \beta f}{\alpha^2 + \beta^2} = \frac{\alpha^3 h + \beta^3 k}{\alpha^2 + \beta^2},$$

or

$$y = -\frac{\alpha}{\beta}x + \frac{h\frac{\alpha^3}{\beta^3} + k}{\frac{\alpha^2}{\beta^2} + 1}$$

$$= mx - \frac{m^3 h - k}{1 + m^2}, \text{ where } m = -\frac{\alpha}{\beta}.$$

7. Tripos 1878. Tuesday morning. No. 5.

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## Paper LXXXI.

1. Let $y = x^{p^{r-1}} = x \,.\, x \,.\, x \,.\, .\, .$ to $p^{r-1}$ factors.

Since $x$ is prime to $p$, $\therefore$ $y$ is prime to $p$

$$\therefore y^{p-1} = 1 + M(p).$$

Now $y^p = \left(x^{p^{r-1}}\right)^p = x^{p^r}$, $\therefore y^{p-1} = x^{p^r - p^{r-1}}$.

2. Let $C$ be the centre of the circle, and produce $CO$ to meet the circumference in $A$. Let $P_1OA = \alpha$, $P_1OP_2 = \beta = P_2OP_3$, &c.,

Then $n\beta = 2\pi$.

$$CP_1^2 = CO^2 + OP_1^2 - 2CO \cdot OP_1 \cos COP_1.$$

$$\therefore a^2 = b^2 + OP_1^2 + 2b \cdot OP_1 \cos \alpha.$$

$$\therefore (a^2 - b^2)\frac{1}{OP_1} = OP_1 + 2b \cos \alpha.$$

$$\text{So } (a^2 - b^2)\frac{1}{OP_2} = OP_2 + 2b \cos(\alpha + \beta), \text{ &c.,}$$

$$\therefore OP_1 + OP_2 + \ldots + OP_n = (a^2 - b^2)\left(\frac{1}{OP_1} + \frac{1}{OP_2} + \ldots \right.$$

$$\left. \ldots + \frac{1}{OP_n}\right) + 2b \cdot S,$$

where $S = \cos \alpha + \cos(\alpha + \beta) + \ldots + \cos(\alpha + \overline{n-1}\beta)$

$$= \sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}\operatorname{cosec}\frac{\beta}{2}$$

$$= 0, \text{ since } \sin\frac{n\beta}{2} = \sin\pi = 0.$$

3. Produce $AO$ to meet the circle round $ABC$ in $D$. Make the angle $AFG = ADB$, and $AEH = ADC$. Then circles will go round $BDGF$, $CDHE$. Then $AGF = ABD$ = sup. of $ACD$ = sup. of $AHE$. $\therefore FG$ is parallel to $HE$. $\therefore AH = OG$. $\therefore BA \cdot AF = DA \cdot AG$,

and $$CA \cdot AE = DA \cdot AH = DA \cdot OG.$$

$$\therefore BA \cdot AF + CA \cdot AE = DA(AG + GO) = DA \cdot AO = AO^2 + AO \cdot OD$$

$$= AO^2 + BO \cdot OC.$$

4. If we project the ellipse into a circle, the inscribed triangle is equilateral, and the sum of the squares on the sides is equal to $\frac{3}{2}$ × sum of the squares on two diameters at right angles. In the ellipse the diameters corresponding to these are conjugate, and the sum of the squares on two conjugate diameters is equal to the sum of the squares on the axes; hence the theorem.

5. Draw $PN$ and $RM$ perpendicular to the axis, and $RZ'$ perpendicular to $AZ$, and let the normal meet the axis in $G$.

Then $$AM : AS :: RZ : ZS$$
$$:: RP : PG$$
$$:: MN : NG.$$

Now $NG = 2AS$, $\therefore MN = 2AM$, $\therefore PZ = RZ'$.

6. Let the first resultant make an angle $\phi$ with the direction of $P$. Then in the first case

$$R^2 = P^2 + Q^2 + 2PQ\cos a \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and $$\frac{P}{\sin(a-\phi)} = \frac{Q}{\sin\phi} = \frac{R}{\sin a} \quad . \quad . \quad . \quad . \quad (2)$$

In the second case,

$$\frac{P+R}{\sin(a-\theta-\phi)} = \frac{Q+R}{\sin(\theta+\phi)} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (2), $\frac{P}{R}\sin a = \sin(a-\phi)$

$$= \sin a\cos\phi - \cos a \cdot \frac{Q}{R}\sin a.$$

$$\therefore \cos\phi = \frac{P + Q\cos a}{R},$$

$$\therefore \cos(a-\phi) = \cos a\cos\phi + \sin a\sin\phi$$

$$= \frac{P\cos a + Q}{R}.$$

From (3), $(P+R)(\sin\theta\cos\phi + \cos\theta\sin\phi)$

$$= (Q+R)\{\sin(a-\phi)\cos\theta - \cos(a-\phi)\sin\theta\}$$

$$\therefore \frac{\sin\theta}{R}\{(P+R)(P+Q\cos a) + (Q+R)(P\cos a + Q)\}$$

$$= \frac{\cos\theta}{R}\{(Q+R)P\sin a - (P+R)Q\sin a\}$$

$$\therefore \tan\theta = \frac{R(P-Q)\sin a}{R\{P+Q+(P+Q)\cos a\} + P^2 + Q^2 + 2PQ\cos a}$$

$$= \frac{(P-Q)\sin a}{P+Q+R+(P+Q)\cos a} \text{ by (1).}$$

7. Let $A$ be the highest, $O$ the lowest point of the vertical diameter, $B$ any point on the circumference. Draw $BD$ perpendicular to $AO$, and make the angle $BOP = BOA$, producing $AB$ to meet $OP$ in $P$. Draw $PK$ perpendicular to $BD$, and $BS$ perpendicular to $OP$.

Then since the velocity at $O$ is due to height $OD$, $BD$ is the directrix of the path described. Since $BO$ is a tangent, the focus is in $OP$. Since $OS = OD$, $S$ is the focus.

The triangles $OBP$, $OBA$ are equal in all respects, as are also $BKP$ and $ABD$; and $OS = OD$, $\therefore SP = DA = BK$, and the angle $SPB = KPB$. $\therefore P$ is a point on the parabola, and $BP$ is a tangent

Since $OP = OA$ = const. the locus of $P$ is a circle, centre $O$ and radius equal to the diameter of the given circle.

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## Paper LXXXII.

1 $$2^n = (1+1)^n = 1 + \left\{n + \frac{n(n-1)}{\lfloor 2}\right\} + \left\{\frac{n(n-1)(n-2)}{\lfloor 3} + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\right\} + \ldots$$

$$= 1 + n\left(1 + \frac{n-1}{2}\right) + \frac{n(n-1)(n-2)}{\lfloor 4}(4 + n - 3) + \ldots$$

$$= 1 + \frac{(n+1)n}{\lfloor 2} + \frac{(n+1)n(n-1)(n-2)}{\lfloor 4} + \ldots$$

2. Let $A$, $B$, $C$, $D$, $E$ be consecutive angular points of the polygon, and let $BD$ meet $AC$ in $M$ and $EC$ in $N$. Then the angle $ACB = BAC = BDC = CBD$. $\therefore MB = MC$. Now the angle $BCD = \frac{n-2}{n}\pi$, $\therefore MBC = MCB = \frac{\pi}{n}$. Similarly $DCN = NDC = \frac{\pi}{n}$. $\therefore$ the triangles $BCM$, $CDN$ are similar; and $BC = CD$, $\therefore$ the triangles are equal in all respects. $\therefore$ if $a$ denote the length of $AB$, the required area = area of polygon $- n \times$ triangle $BCM$

$$= \frac{na^2}{4}\cot\frac{\pi}{n} - \frac{na^2 \sin^2\frac{\pi}{n}}{2\sin\frac{n-2}{n}\pi}$$

$$= \frac{na^2}{4}\cot\frac{\pi}{n} - \frac{na^2}{2}\cdot\frac{\sin^2\frac{\pi}{n}}{\sin\frac{2\pi}{n}}$$

$$= \frac{na^2}{4}\left(\cot\frac{\pi}{n} - \tan\frac{\pi}{n}\right)$$

$$= \frac{na^2}{2}\cot\frac{2\pi}{n}.$$

3. Draw $AD$ perpendicular to $BC$. Then circles will go round $AFDC$ and $AEDB$.

$$\therefore BF.BA = BD.BC, \text{ and } CE.CA = CD.CB$$

$$\therefore AB.BF + AC.CE = BC(BD + DC) = BC^2,$$

4. Since the angle $BPC$ is the supplement of $A$, $\therefore$ the circles round $PBC$ and $PAC$ are each equal to the circle round $ABC$.

$$\therefore A'P = B'P = C'P = R.$$

$\therefore$ $P$ is the centre of the circle round $A'B'C'$.

Since $A'P = A'C$ and $B'P = B'C$, we have two isosceles triangles on the same base $PC$, $\therefore$ $PC$ is perpendicular to $A'B'$. $\therefore$ $A'B'$ is parallel to $AB$. Similarly it may be shewn that $B'C'$ is parallel to $BC$, and $C'A'$ to $CA$. $\therefore$ $A'O$, which is perpendicular to $BC$, is also at right angles to $B'C'$. $\therefore$ $O$ is the orthocentre of $A'B'C'$.

Since the circles, centres $A'$ and $C'$, are each equal to the circle round $ABC$, $\therefore$ $CA' = CB' = CO$. $\therefore$ $C$ is the centre of the circle round $A'OB'$. Similarly it may be shewn that $A$ and $B$ are the centres of the circles round $B'OC'$, $C'OA'$.

Consider the triangle $BPC$. Its orthocentre is $A$, and the centre of its circum-circle is $A'$. Since $AO = A'P$, $\therefore$ $AA'$ bisects $OP$. $\therefore$ the centre of its nine-point circle, which is at the middle point of $AA'$, is also at the middle point of $OP$, and the radius $= \frac{1}{2}A'P = \frac{1}{2}R$.

Consider the triangle $B'OC'$. Its orthocentre is $A'$, and the centre of its circum-circle is $A$. $\therefore$ the centre of its nine-point circle is at the middle point of $OP$, and the radius $= \frac{1}{2}AO = \frac{1}{2}R$.

Consider $A'B'C'$. Its orthocentre is $O$, and the centre of its circum-circle is $P$, and radius $= A'P = R$.

$\therefore$ the 8 triangles have the same nine-point circle.

5. In general five conditions are required to determine a conic. Since the focus is the pole of the directrix, the focus being given is equivalent to two conditions, and only one ellipse can be described having its focus at a given point and touching the sides of a triangle.

Let $ABC$ be a triangle, $P$ the orthocentre, $O$ and $N$ the centres of the circum- and nine-point circles. Then the angle $OAB = PAC$, each being the complement of $ACB$. $\therefore$ since $AB$ and $AC$ are tangents, we see that if either $O$ or $P$ is a focus, the other is also, and the centre is at $N$, which is the middle point of $OP$. Again, since the centre is the pole of the line at infinity, the centre being given is equivalent to two conditions, and $\therefore$ only one ellipse can be described touching the sides of a triangle, and having its centre at the centre of the nine-point circle. $\therefore$ by the converse of the first case, the foci are at $O$ and $P$.

6. Let $O$, $O'$ be the centres of the small and large circles, $C$ and $D$ their points of contact with the beam, $A$ and $B$ the points where they touch the plane. Produce $BA$ and $DC$ to meet in $H$. Then $AHC = \alpha$. Since the circle $O$ is in equilibrium, and is acted on by forces at $A$ and $C$, $\therefore$ the resultant force on the cylinder acts at $C$ along $CA$. $\therefore$ the resultant force on the beam at $C$ acts along $AC$. Similarly the resultant force on the beam at $D$ acts along $BD$, which is parallel to $CA$. $\therefore$ if $E$ be the middle point of the beam, $EF$ perpendicular to $AB$, and $EG$ parallel to $AC$ or $BD$, the resultant of $P$ and $W$ acts along $EG$.

$$\therefore \frac{P}{W} = \frac{\sin FEG}{\sin GEP} = \frac{\sin FEG}{\sin EGB} = \frac{\sin FEG}{\sin FGE} = \tan FEG = \tan\frac{\alpha}{2}$$

for $2EGF = \pi - \alpha$.

7. Let the tangent at $P$ meet the tangent at $A$ in $Y$. Join $SY$ cutting the curve in $Q$. Then $SY$ bisects the angle $ASP$. $\therefore$ $ASY = 60°$. Draw the ordinate $PM$, and join $AP$. Then $PM = 2AY$. The parabolic area $AQP = \frac{2}{3}$ the triangle $AYP$. (Bes. *Con.* p. 162.)

$\therefore$ parabolic area $ASP$ = parabolic area $AQP$ + triangle $ASP$

$$= \tfrac{2}{3}\, AYP + ASP$$

$$= \tfrac{1}{3}\, AY \,.\, YM + \tfrac{1}{2}\, AS \,.\, PM.$$

Now $$PM = 2AY = 2AS \tan 60°,$$

$$AM = \frac{PM^2}{4AS} = AS \,.\, \tan^2 60°,$$

$\therefore$ parabolic area $ASP = AS^2 (\tan 60° + \frac{1}{3} \tan^3 60°) = AS^2\, 2\sqrt{3}$.

So „ $ASp = AS^2 (\tan 30° + \frac{1}{3} \tan^3 30°) = AS^2 \cdot \dfrac{10}{9\sqrt{3}}$.

And time in $AP$ : time in $Ap$ :: area $ASP$ : $ASp$

$$:: 2\sqrt{3} : \frac{10}{9\sqrt{3}}$$

$$:: 27 : 5.$$

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## Paper LXXXIII.

1. Let $x$ denote the rateable value of the University. Then $\frac{x}{6}$ is the amount it pays. $\therefore$ the amount paid by the town $= \frac{x}{2}$, and the rateable value of the town $= 8x$, $\therefore$ the rateable values are as 1 : 8.

The total amount of rates paid $= \frac{x}{2} + \frac{x}{6} = \frac{2x}{3}$.

The total value of rateable property $= x + 8x = 9x$,

$\therefore$ the average rate is $£\frac{2}{27}$ in the £.

$\therefore$ in the £ the University would save $£\left(\frac{1}{6} - \frac{2}{27}\right) = \frac{1}{6} \cdot \frac{5}{9} = \frac{5}{9}$ of their present payment.

2. $$(1) = \tfrac{1}{30}(6n^5 + 15n^4 + 10n^3 - n).$$

By trial we find that this is integral when $n = 1, 2, 3$. Suppose it is integral when $n = p$. Then writing $p + 1$ for $n$ in the expression and subtracting its value when $n = p$, we have

$\frac{1}{30}\{30p^4 + 120p^3 + 180p^2 + 120p + 30\}$, which is integral.

$\therefore$ if the expression is integral for any value of $n$, it is also integral for the next consecutive value of $n$. Now we know it is an integer when $n = 3$, $\therefore$ it is an integer when $n = 4$, $\therefore$ by induction it is always integral.

$$(2) = \tfrac{1}{12}(2n^6 + 6n^5 + 5n^4 - n^2).$$

By trial this is integral when $n = 1, 2, 3$. Suppose it is integral when $n = p$, then as in (1), the difference of its values when $n = p$ and $n = p + 1$ is

$\frac{1}{12}(12p^6 + 60p^5 + 120p^3 + 120p^2 + 60p - 12)$, which is integral.

$\therefore$ as before, the proof follows by induction.

3. Let $O$ be the point of intersection, $G$, $H$, $K$ the feet of the perpendiculars on $BC$, $CA$, $AB$.

Then $GK = OK \dfrac{\sin GOK}{\sin OGK} = OB \sin B = \frac{2}{3} BE \sin B$.

Similarly $\quad GH = \frac{2}{3} CF \sin C$; $\quad FH = \frac{2}{3} AD \sin A$.

Also $OK = OB \sin OBK = \frac{2}{3} EB . \sin EBK = \frac{1}{3} b \sin A = \frac{1}{3} a \sin B$.

Similarly $OH = \frac{1}{3} c \sin A = \frac{1}{3} a \sin C$; $OG = \frac{1}{3} b \sin C = \frac{1}{3} c \sin B$.

$\therefore$ area of triangle $GHK = \frac{1}{18}(a^2 + b^2 + c^2) \sin A \sin B \sin C$.

$$\therefore R = \frac{FG . GH . HF}{4 . \triangle FGH} = \frac{8}{27} \frac{AD . BE . CF}{4 . \triangle FGH} \sin A \sin B \sin C$$

$$= \frac{4}{3} \frac{AD . BE . CF}{a^2 + b^2 + c^2}.$$

4. Let the chord through $S$ cut the inner curve in $P$, $p$ and the outer in $P'$, $p'$. Let the tangents at $P$, $p$ meet in $R$, those at $P'$, $p'$ in $R'$. Produce $PR$ to meet $p'R'$ in $T$, and produce $pR$ to meet $P'R'$ in $T'$. Then since tangents at the extremities of a focal chord intersect at right angles on the directrix, $\therefore RTR'T'$ is a rectangle. Since $SR$ and $SR'$ are perpendicular to the chord, $\therefore RR'$ passes through $S$. Let $TT'$ and $RR'$ meet in $F$, and draw $P'K'$ perpendicular to the directrix of the outer parabola.

Then the angle $FT'R' = T'R'F = T'R'K'$. $\therefore TT'$ is parallel to the directrix.

5. Let the tangents at $A$ and $B$ meet in $T$. Draw $PD$, $PE$, $PF$ perpendiculars on $TA$, $TB$, $AB$. Similarly draw $QD'$, $QE'$, $QF'$ perpendiculars on the same lines. Then by Casey, p. 33,

$$PF^2 = PD . PE; \quad QF'^2 = QD' . QE'. \quad \text{And } PD = QD'.$$

$$\therefore \frac{PE}{QE'} = \frac{PF^2}{QF'^2} = \frac{PR^2}{QR^2} \text{ from similar triangles.}$$

Similarly, if $PG$, $QG'$ be perpendiculars on the tangent at $C$, we shall have $\dfrac{PG}{QG'} = \dfrac{PS^2}{QS^2}$.

$$\therefore \left(\frac{PR . PS}{QR . QS}\right)^2 = \frac{PE . PG}{QE' . QG'} = \frac{PM^2}{QN^2}.$$

*6. If two heavy particles be projected in the same vertical plane at the same instant from two given points $A$, $B$, with the same velocity $u$ in directions $AQ$, $BQ$ and meet at the point $P$ after a time $t$, then if $PQ$ be drawn vertically upwards and equal to $\frac{1}{2}gt^2$, it follows that if gravity did not act the particles would pass through $Q$ at the same instant. This requires $AQ = ut = BQ$. $\therefore$ the directions of projection, viz. $AQ$, $BQ$ are equally inclined to $AB$, and $\therefore$ the sum of their inclinations to the horizon is const. Again, let $A$, $B$ be the two points of projection, and let $AB = 2a$. Bisect $AB$ in $E$, and let $Q$ be any point in the line through $E$ at right angles to $AB$. Then if $u$ be the velocity of projection, $AQ = ut = BQ$. Draw $QP$ vertically downwards and $= \frac{1}{2}gt^2$. Then we require to find the locus of $P$ as $Q$ moves along the fixed line $EQ$.

Through $E$ draw $EN$ parallel to $QP$ and $PN$ parallel to $QE$.

Then $QPNE$ is a parallelogram.

Then $AQ = ut$, $\therefore AQ \propto t$ since $u$ is constant.

And $QP = \frac{1}{2}gt^2$, $\therefore QP \propto t^2$,

$\therefore AQ^2 \propto QP$, $\therefore AQ^2 = \lambda QP$ where $\lambda$ is some constant.

On $EN$ take a point $O$ such that $\lambda . OE = AE^2$. Then $O$ is a fixed point.

$$\therefore AQ^2 - AE^2 = \lambda(QP - OE),$$

$$\therefore QE^2 = PN^2 = \lambda . ON.$$

$\therefore$ the locus of $P$ is a parabola having $ON$ for a diameter, *i.e.* a vertical line through the middle point of $AB$. The ordinates to that diameter are perpendicular to $AB$.

7. Draw $AD$ at right angles to $BC$. Since the resultant passes through the circum-centre and the orthocentre, the sum of the moments of the forces about each of these points will vanish.

If $\rho$ be the radius of the circum-circle, the perpendicular from the circum-centre on $BC = \rho \cos A$.

The perpendicular from the orthocentre on $EC = BD \cot C = c . \cos B \cot C$,

$$= \frac{c}{\sin B} \cos B \cos C.$$

$$\therefore P \cos A + Q \cos B + R \cos C = 0. \quad . \quad . \quad . \quad . \quad (1)$$

$$P \cos B \cos C + Q \cos C \cos A + R \cos A \cos B = 0. \quad . \quad . \quad (2)$$

From (1) and (2) we at once obtain the required result.

## Paper LXXXIV.

1. This may be shewn by ordinary multiplication, or may be proved as follows.

Denote the given expression by $A$.

If for $b$ we write $a$, $A = 0$. $\therefore a - b$ is a factor of $A$. Similarly $a - c$, $a - d$, $b - c$, $b - d$, $c - d$ are factors. Now $A$ is symmetrical with respect to $a$, $b$ and $c$, and is of the 7th degree.

$$\therefore A \equiv (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)\{k(a+b+c)+ld\},$$

where $k$ and $l$ are numerical constants. Now on the right we have the term $ka^4b^2c$, whilst $A$ does not contain any term which has one of its factors raised to the 4th degree. $\therefore k = 0$. And $d$ is not a factor of $A$. $\therefore l = 0$.

2. Let $x = \dfrac{a}{b+x}$. $\therefore x^2 + bx - a = 0$. $\therefore x = \dfrac{-b \pm \sqrt{b^2+4a}}{2}$.

We must obviously take the $+^{\text{ve}}$ sign. $\therefore x^2 = \dfrac{b^2+2a-b\sqrt{b^2+4a}}{2}$.

Let $y = \dfrac{a^2}{2a+b^2-y}$. $\therefore y^2-(2a+b^2)y+a^2=0$. $\therefore y=\dfrac{b^2+2a\pm b\sqrt{b^2+4a}}{2}$.

$\therefore$ taking the $-^{\text{ve}}$ sign, we have $y = x^2$.

3. Let $D$ be the middle point of the base, and $E$ the foot of the perpendicular from $A$ on $BC$. Produce $AD$, $AE$ to meet the circumcircle in $D'$, $E'$. Then if $p = AD'$, $u = AE'$,

$$\frac{p}{AB} = \frac{\sin ABD'}{\sin AD'B} = \frac{\sin\left(B+\frac{A}{2}\right)}{\sin C} = \frac{\cos\frac{B-C}{2}}{\sin C}. \quad \therefore p = \frac{a}{\sin A}\cos\frac{B-C}{2}.$$

$$\frac{u}{AC} = \frac{\sin ACE'}{\sin AE'C} = \frac{\sin\left\{\frac{\pi}{2}-(B-C)\right\}}{\sin B} = \frac{\cos(B-C)}{\sin B}. \quad \therefore u = \frac{a}{\sin A}\cos(B-C).$$

From these we can write down the values of $q$, $r$, $v$, $w$.

$$\therefore p^2(w-v) = \frac{a^3}{\sin^3 A}\cos^2(B-C)\{\cos(A-B)-\cos(C-A)\}$$

$$= \frac{a^3}{\sin^3 A}\left\{1 + \cos(B - C)\right\}\left\{\cos(A - B) - \cos(C - A)\right\}$$

$$= \frac{a^3}{\sin^3 A}(1 + C_1)(C_2 - C_3),$$

where $C_1 = \cos(B - C)$, $C_2 = \cos(A - B)$, $C_3 = \cos(C - A)$.

Similarly $q^2(u - w) = \frac{a^3}{\sin^3 A}(1 + C_2)(C_3 - C_1)$,

$$r^2(v - u) = \frac{a^3}{\sin^3 A}(1 + C_3)(C_1 - C_2),$$

$$\therefore p^2(w - v) + q^2(u - w) + r^2(v - u) = 0.$$

4. Since $ad$ is parallel to $BC$, $\therefore$ arc $Cd$ = arc $Ba$ = arc $Ab$, for $AOa$, $BOb$ are diameters. $\therefore$ the angle $dbC = ACb$. $\therefore$ $Eb = EC = Ef$, since $bCf$ is a right angle, being in the semi-circle $bCB$. $\therefore$ $E$ is the middle point of $bf$; and $O$ is the middle point of $bB$. $\therefore$ $OE$ is parallel to $BC$.

5. Let $S$ be the common focus, $CL$ one of the asymptotes. Draw $SD$ perpendicular to $CL$ and $DX$ perpendicular to the axis. Then since $CL$ is a tangent to $P$, $\therefore$ $D$ is on the tangent at the vertex of $P$. But since $CL$ is an asymptote, $\therefore$ $D$ is on the directrix of $H$.

Let $R$ be one of the points of intersection, and let the tangent to $P$ at $R$ meet the axis in $A'$. Draw the ordinate $RN$, and let $A$ be the vertex of $H$ nearer $S$.

Then since $R$ is on $P$, $\therefore$ $SR = SA'$, and $A'X = NX$.

Since $R$ is on $H$, $\therefore$ $SR : NX :: SA : AX$

$$\therefore SA' : A'X :: SA : AX.$$

$\therefore$ $A'$ is the further vertex of $H$.

6. If $C$ be the centre, $TC$ bisects $PQ$ and is $\therefore$ the direction of the resultant of the forces $TP$, $TQ$. $\therefore$ if $C$ be fixed the ellipse will remain at rest.

7. The equation to the normal at $P$ is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

If $\theta$ be the eccentric angle of the point $Q$ where $(A)$ meets the ellipse, the point $(a\cos\theta, b\sin\theta)$ will lie on $(A)$.

$$\therefore \frac{a^2 \cos\theta}{\cos\phi} - \frac{b^2 \sin\theta}{\sin\phi} = a^2 - b^2,$$

$$\therefore a^2\left(1 - \frac{\cos\theta}{\cos\phi}\right) = b^2\left(1 - \frac{\sin\theta}{\sin\phi}\right),$$

$$\therefore a^2 \tan\phi = b^2 \frac{\sin\phi - \sin\theta}{\cos\phi - \cos\theta} = -b^2 \cot\tfrac{1}{2}(\theta + \phi),$$

$$\therefore a^2 \tan\phi + b^2 \cot\tfrac{1}{2}(\theta + \phi) = 0.$$

Let $(\xi, \eta)$ be the coordinates of the middle point of $PQ$.

Then $$\xi = \frac{a}{2}(\cos\theta + \cos\phi) = a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

$$\eta = \frac{b}{2}(\sin\theta + \sin\phi) = b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

$$\therefore \frac{\xi}{\eta} = \frac{a}{b}\cot\frac{\theta+\phi}{2} = -\frac{a^3}{b^3}\tan\phi,$$

$$\therefore \frac{\cos\phi}{a^3\eta} = \frac{\sin\phi}{-b^3\xi} = \frac{1}{\sqrt{a^6\eta^2 + b^6\xi^2}},$$

$\therefore$ substituting in $(A)$, we have

$$\frac{a\xi\sqrt{b^6\xi^2 + a^6\eta^2}}{a^3\eta} + \frac{b\eta\sqrt{b^6\xi^2 + a^6\eta^2}}{b^3\xi} = a^2 - b^2,$$

$$\therefore (b^6\xi^2 + a^6\eta^2)(b^2\xi^2 + a^2\eta^2)^2 = a^4b^4\xi^2\eta^2(a^2 - b^2)^2$$

or $$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2\left(\frac{x^2}{a^6} + \frac{y^2}{b^6}\right) = \frac{(a^2 - b^2)^2}{a^6b^6}x^2y^2.$$

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## Paper LXXXV.

1. Let $x$, $x + 1$, $y$, $y + 1$ denote the digits.

Then $10x + x + 1 = y(y + 1)$, $\therefore 11x + 1 = y^2 + y$ . . (1)

and $100x + 10y + x + 1 = 9(y+1)^2$, $\therefore 101x + 10y + 1 = 9(y^2 + 2y + 1)$ (2)

By (1) $$99x + 9 = 9y^2 + 9y,$$

By (2) $$101x + 1 = 9y^2 + 8y + 9,$$

$$\therefore 2x + y = 17; \quad \therefore y = 17 - 2x,$$

$\therefore$ from (1) $11x + 1 = (17 - 2x)(18 - 2x) = 306 - 70x + 4x^2$,

$$\therefore 4x^2 - 81x + 305 = 0, \quad \therefore x = 5 \text{ or } 15\tfrac{1}{4}.$$

Taking the integral value, $y = 17 - 10 = 7$,

$\therefore$ the numbers are 56, 78.

2. Let $\alpha, \beta, \gamma, \delta$ be the vertices of the triangles described on the bases $AB$, $BC$, $CD$, $DA$ respectively.

Produce $\alpha A$ and $\gamma D$ to meet in $X$.

Then $XA\delta = \frac{\pi}{2} - A$; $\delta AD = \frac{\pi}{4}$; $XAD = \frac{3\pi}{4} - A$.

Similarly $XDA = \frac{3\pi}{4} - D$; $\therefore AXD = A + D - \frac{\pi}{2}$.

$$\alpha\gamma^2 = aX^2 + \gamma X^2 - 2aX \cdot \gamma X \cos X$$

$$= \left(\frac{a}{\sqrt{2}} + AX\right)^2 + \left(\frac{c}{\sqrt{2}} + DX\right)^2 - 2\left(\frac{a}{\sqrt{2}} + AX\right)\left(\frac{c}{\sqrt{2}} + DX\right)\cos X$$

$$= \frac{a^2}{2} + \frac{c^2}{2} - ac \sin(A + D) + AX^2 + DX^2 - 2AX \cdot XD \cos X$$
$$+ \sqrt{2}a(AX - DX\cos X) + \sqrt{2}c(DX - AX\cos X)$$

$$= \frac{a^2}{2} + \frac{c^2}{2} - ac\sin(A + D) + d^2 + \sqrt{2}ad\cos DAX + \sqrt{2}cd\cos ADX$$

$$= \frac{a^2}{2} + \frac{c^2}{2} - ac\sin(A+D) + d^2 + \sqrt{2}ad\cos\left(\frac{\pi}{4} + A\right) - \sqrt{2}cd\cos\left(\frac{\pi}{4} + D\right)$$

$$= \frac{a^2}{2} + \frac{c^2}{2} - ac\sin(A+D) + d^2 + ad(\sin A - \cos A) + cd(\sin D - \cos D).$$

Similarly by producing $\alpha B$, $\gamma C$ to meet in $Y$, we get

$$\alpha\gamma^2 = \frac{a^2}{2} + \frac{c^2}{2} - ac\sin(B+C) + b^2 + bc(\sin C - \cos C) + ab(\sin B - \cos B),$$

$$\therefore 2\alpha\gamma^2 = a^2 + b^2 + c^2 + d^2 + da(\sin A - \cos A) + ab(\sin B - \cos B)$$
$$+ bc(\sin C - \cos C) + cd(\sin D - \cos D),$$

and from symmetry, $2\beta\delta^2$ is equal to the same quantity.

3. On $BC$, $CA$, $AB$ describe the squares $BCLK$, $CANM$, $ABHG$, and join $GN$, $ML$, $KH$.

Then $GN^2 = GA^2 + AN^2 - 2GA\ AN \cos GAN$

$= b^2 + c^2 + 2bc \cos A = 2(b^2 + c^2) - a^2.$

Now if $S$ denote the area of a triangle whose sides are $a, b, c$,

$$\tfrac{1}{16} S^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$$
$$= (a^2 + b^2 + c^2)^2 - 2(a^4 + b^4 + c^4),$$

$\therefore$ if $S'$ denote the area of the triangle $DEF$ whose sides are $GN, ML, HK$,

$$\tfrac{1}{16} S'^2 = (GN^2 + ML^2 + HK^2)^2 - 2(GN^4 + ML^4 + HK^4)$$
$$= 9(a^2 + b^2 + c^2)^2 - 2\left\{4(b^2 + c^2)^2 + a^4 - 4a^2(b^2 + c^2)\right\}$$
$$- 2\left\{4(c^2 + a^2)^2 + b^4 - 4b^2(c^2 + a^2)\right\}$$
$$- 2\left\{4(a^2 + b^2)^2 + c^4 - 4c^2(a^2 + b^2)\right\}$$
$$= 9(a^2 + b^2 + c^2)^2 - 18(a^4 + b^4 + c^4)$$
$$= 9\left\{(a^2 + b^2 + c^2)^2 - 2(a^4 + b^4 + c^4)\right\} = 9 \cdot \tfrac{1}{16} S^2.$$

$\therefore S' = 3S.$

Again, if $G'N'$ be a side of the triangle formed as $DEF$ was,

$$G'N'^2 = 2(HK^2 + LM^2) - GN^2$$
$$= 4(c^2 + a^2) - 2b^2 + 4(a^2 + b^2) - 2c^2 - 2b^2 - 2c^2 + a^2$$
$$= 9a^2.$$

$\therefore G'N' = 3a$, &c.

4. Tripos 1878. Wednesday morning. No. 8.

5. Let $PCP'$ be the diameter down which the time of descent is a minimum, and let $DCD'$ be the diameter conjugate to $PCP'$. Then if a circle be described touching the transverse axis at $C$ and the hyperbola, its point of contact will be $P$. Let $PT$ be the common tangent at $P$. Then $TC = TP$.

Let $x'y'$ be the coordinates of $P$. Then the equation to $PT$ is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1,$$

$\therefore$ at $T$, $y = 0$, $x = \dfrac{a^2}{x'}$. $\therefore$ expressing the fact that $CT = TP$,

$$\frac{a^4}{x'^2} = y'^2 + \left(x' - \frac{a^2}{x'}\right)^2 = x'^2 + y'^2 - 2a^2 + \frac{a^4}{x'^2},$$

$$\therefore 2a^2 = x'^2 + y'^2 = CP^2.$$

Now $CP^2 - CD^2 = a^2 - b^2$, $\therefore CD^2 = a^2 + b^2 = CS^2$, $\therefore DD' = SS'$.

6. The equation to the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$\therefore$ writing $r\cos\theta$ for $x$, and $r\sin\theta$ for $y$,

$$r^2 \frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2} = 1,$$

$$\therefore r^2 = \frac{a^2b^2}{b^2 + (a^2 - b^2)\sin^2\theta} = \frac{a^2}{1 + e^2\sin^2\theta},$$

$$\therefore r = a(1 + e^2\sin^2\theta)^{-\frac{1}{2}}$$
$$= a\{1 - \tfrac{1}{2}e^2\sin^2\theta + \text{terms involving higher powers of } e\}$$
$$= a(1 - \tfrac{1}{2}e^2\sin^2\theta) \text{ nearly.}$$

7. Employing the previous question, we have

$$a = 4000; \quad e = \cdot 08, \quad \therefore e^2 = \cdot 0064.$$

Denoting the distance of the mouth from the earth's centre by $\rho_1$, and the distance of the source by $\rho_2$,

$$\rho_1 = 4000\{1 - \tfrac{1}{2}(\cdot 0064)\sin^2 30^\circ\}$$
$$= 4000\{1 - \tfrac{1}{2}(\cdot 0064).\tfrac{1}{4}\},$$
$$\rho_2 = 4000\{1 - \tfrac{1}{2}(\cdot 0064)\sin^2 45\}$$
$$= 4000\{1 - \tfrac{1}{2}(\cdot 0064)\tfrac{1}{2}\},$$
$$\therefore \rho_1 - \rho_2 = 4000 \times \tfrac{1}{8}(\cdot 0064) = 4000 \times \cdot 0008 = 3\cdot 2.$$
$$= 3\tfrac{1}{5} = 3\tfrac{1}{4} \text{ nearly.}$$

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## Paper LXXXVI.

1. If $n$ be very great, $n$ events happen in $n$ years, and the chance that any particular one of the events does not happen in the given year is $\frac{n-1}{n}$. Similarly the chance that any other particular one of the events does not happen in the given year is $\frac{n-1}{n}$. $\therefore \left(\frac{n-1}{n}\right)^2$ is the

chance that neither of these events happens in the given year. Similarly the chance that no one of the events happens in the given year

$$= \left(\frac{n-1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n = 1 - \frac{n}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdot$$

$$= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots \text{ when } n \text{ is very great}$$

$$= e^{-1}.$$

2. By Todh. *Trig.* Cap. viii., Ex. 16, and Art. 114,

$$(1) \qquad = 8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \times 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= 4\sin A \sin B \sin C$$

$$= \sin 2A + \sin 2B + \sin 2C.$$

Again, expanding (2), and taking together corresponding terms,

$$(2) = 2\{\cos(A-B) + \cos C + \cos(B-C) + \cos A + \cos(C-A) + \cos B\}$$

$$= 2\{\cos(A-B) - \cos(A+B) + \cos(B-C) - \cos(B+C) + \cos(C-A) - \cos(C+A)\}$$

$$= 4(\sin A \sin B + \sin B \sin C + \sin C \sin A).$$

3. If $rr_1 = r_2r_3$, $\tan^2\frac{A}{2} = \frac{(s-b)(s-c)}{s(s-a)} = \frac{rr_1}{r_2r_3} = 1$. $\therefore A = \frac{\pi}{2}$.

4. First let $D$ and $A$ be on the same side of $BC$. Produce $AD$ and $CB$ through $D$ and $B$ to meet in $E$.

Then the angle $ADB$ = sup. of $ACB$ = sup. of $ABC$ = $ABE$; and $DAB$ is common to the two triangles $DAB$, $ABE$; $\therefore$ they are similar.

$$\therefore DB : DA :: BE : AB. \quad . \quad . \quad . \quad (1)$$

Also the angle $ADC = ABC = ACB$, and $DAC$ is common to the two triangles $ADC$, $ACE$; $\therefore$ they are similar.

$$\therefore DC : DA :: CE : AC \text{ (or } AB). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$\therefore$ from (1) and (2) by subtraction we have

$$DC - DB : DA :: BC : AB, \textit{ i.e.} \text{ in a constant ratio.}$$

Similarly it can be shewn that when $D$ and $A$ are on opposite sides of $BC$,

$$DC + DB : DA :: BC : AB.$$

5. Let the tangents at $Q$ and $Q'$ meet in $T$, and let $CT$, which is one of the equiconjugate diameters, meet $QQ'$ in $V$.

Then since $TQO$, $TQ'O$ are right angles, $\therefore$ $O$, $Q$, $Q'$, $T$ are concyclic.

And $QV^2 = CV \,.\, VT$, $\therefore$ $C$ is on the same circle.

6. Let $CD$ be the diameter conjugate to $CA$. Produce $CA$ to meet the circle in $B$, and draw $CE$, a tangent to the circle.

Then $$CE^2 = CA \,.\, CB = CA^2 + CA \,.\, AB.$$

Now $AB = \frac{1}{2}$ chord of curvature in direction $AC = \dfrac{CD^2}{CA}$.

$\therefore$ $CE^2 = CA^2 + CD^2 = a^2 + b^2$ = square of radius of director circle.

$\therefore$ the two circles cut orthogonally.

7. Let $u$ denote $A$'s velocity before impact; $u'$, $v'$ the velocities of $A$ and $B$ estimated along $AB$ after impact, $u'_1$ the velocity of $A$ measured at right angles to $AB$ after impact. Let $e$ be the elasticity, $m$, $m'$ the masses of $A$ and $B$.

Then $$mu \cos \alpha = mu' + m'v'; \quad eu \cos \alpha = v' - u',$$

$$mu \cos \alpha + meu \cos \alpha = (m + m')\, v',$$

$$\therefore v' = (1 + e) \frac{mu \cos \alpha}{m + m'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore u' = u \cos \alpha \frac{m - em'}{m + m'}, \quad \text{and} \quad u'_1 = u \sin \alpha,$$

$\therefore$ if $V$ be the velocity of $A$ after impact,

$$V^2 = u'^2 + u'_1{}^2 = u^2 \left\{ \sin^2 \alpha + \left( \frac{m - em'}{m + m'} \right)^2 \cos^2 \alpha \right\}. \quad . \quad . \quad (2)$$

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## Paper LXXXVII.

1. In the first line of the question for $b$ read $a + b$.

In the A.P. the common difference $= b$. $\therefore$ $x = a + 2b$.

In the G.P. the common ratio $= \dfrac{a + b}{a}$, $\therefore$ $y = \dfrac{(a + b)^2}{a}$.

In the H.P. the reciprocals of the terms are

$$\frac{1}{a}, \frac{1}{a + b}, \frac{2}{a + b} - \frac{1}{a}, \quad \therefore z = \frac{a(a + b)}{a - b},$$

$$\therefore\ y(x-3z)^2 = \frac{(a+b)^2}{a}\left\{a+2b-\frac{3a(a+b)}{a-b}\right\}^2$$

$$= \frac{4(a+b)^2}{a(a-b)^2}\left\{a^2+ab+b^2\right\}^2$$

$$4z(y^2+xz) = \frac{4a(a+b)}{a-b}\left\{\frac{(a+b)^4}{a^2}+\frac{a(a+b)(a+2b)}{a-b}\right\}$$

$$= \frac{4(a+b)^2}{a(a-b)^2}\left\{2a^4+4a^3b-2ab^3-b^4\right\},$$

$$\therefore\ y(x-3z)^2+4z(y^2+xz) = \frac{4(a+b)^2}{a(a-b)^2}\cdot 3a^2(a+b)^2$$

$$= 12\,\frac{(a+b)^2}{a}\cdot\frac{a^2(a+b)^2}{(a-b)^2} = 12yz^2.$$

Again
$$\frac{1}{n}\left\{a+(a+b)+\ldots+l\right\}^2$$

$$= \frac{n}{4}\left\{4a^2+4ab(n-1)+(n-1)^2b^2\right\}$$

$$= na^2+abn(n-1)+\frac{n(n-1)}{4}b^2.$$

And
$$al+(a+b)(l-b)+\ldots$$

$$= a\left\{a+(n-1)b\right\}+(a+b)\left\{a-b+(n-1)b\right\}+\ldots$$

$$= a^2+a^2-b^2+a^2-4b^2+\ldots \text{ to } n \text{ terms}$$

$$+(n-1)b\left\{a+(a+b)+(a+2b)+\ldots \text{ to } n \text{ terms}\right\}$$

$$= na^2-b^2(1^2+2^2+\ldots \text{to } n-1 \text{ terms})+\frac{n(n-1)}{2}b\left\{2a+(n-1)b\right\}$$

$$= na^2+abn(n-1)+b^2\cdot\frac{n(n-1)}{2}\left(n-1-\frac{2n-1}{3}\right)$$

$$\therefore \text{ given expression}$$

$$= \frac{n(n-1)}{12}b^2\left\{3(n-1)-6(n-1)+2(2n-1)\right\}$$

$$= \frac{n(n^2-1)b^2}{12}.$$

2. $$2\sqrt{6}+1=\sqrt{25+4\sqrt{6}}=5\left(1+\frac{2^2\sqrt{6}}{5^2}\right)^{\frac{1}{2}}$$

$$=5\left(1+\frac{1}{2}\cdot\frac{2^2 6^{\frac{1}{2}}}{5^2}-\frac{1}{2!}\cdot\frac{2^3.3}{5^4}+\frac{1.3}{3!}\cdot\frac{2^3.6^{\frac{3}{2}}}{5^6}-\frac{1.3.5}{4!}\cdot\frac{2^6.3^2}{5^8}+\ldots\right),$$

So $2\sqrt{6}-1$

$$=5\left(1-\frac{1}{2}\cdot\frac{2^2 6^{\frac{1}{2}}}{5^2}-\frac{1}{2!}\cdot\frac{2^3.3}{5^4}-\frac{1.3}{3!}\cdot\frac{2^3.6^{\frac{3}{2}}}{5^6}-\frac{1.3.5}{4!}\cdot\frac{2^6.3^2}{5^8}-\ldots\right),$$

$$\therefore 4\sqrt{6}=2.5\left(1-\frac{1}{2!}\cdot\frac{2^3.3}{5^4}-\frac{1.3.5}{4!}\cdot\frac{2^6.3^2}{5^8}-\ldots\right).$$

3. $2\{\sin(a+2\beta)\sin(a+3\beta)+\sin(a+3\beta)\sin a+\sin a\sin(a+\beta)\}$

$=\cos\beta-\cos(2a+5\beta)+\cos 3\beta-\cos(2a+3\beta)+\cos\beta-\cos(2a+\beta)$

$=\cos\beta+2\cos 2\beta\cos\beta-\cos(2a+3\beta)(1+2\cos 2\beta)$

$=(1+2\cos 2\beta)\{\cos\beta-\cos(2a+3\beta)\}$

$=2(1+2\cos 2\beta)\sin(a+\beta)\sin(a+2\beta)$,

$\therefore$ the given expression

$$=\frac{\sin\beta(1+2\cos 2\beta)}{\sin(a+3\beta)}=\frac{\sin\beta+\sin 3\beta-\sin\beta}{\sin(a+3\beta)}=\frac{\sin 3\beta}{\sin(a+3\beta)}.$$

4. Let $n$ be the number of sides in the one whose angles are measured in degrees, $m$ the number in the other.

Then if $D$ be the number of degrees in an angle of the 1st, $G$ the number of grades in an angle of the 2nd,

$$D=\frac{2(n-2)}{n}.90\,;\quad G=\frac{2(m-2)}{m}.100\,;\quad \text{and}\quad D=G.$$

$$\therefore mn=20n-18m,$$

$$\therefore m=\frac{20n}{n+18}=20\,\frac{n+18-18}{n+18}=20-\frac{360}{n+18},$$

$\therefore n+18$ is a divisor of 360.

Now $360=1\times 360=2\times 180=3\times 120=4\times 90=5\times 72=6\times 60$

$=8\times 45=9\times 40=10\times 36=12\times 30=15\times 24=18\times 20,$

$\therefore n+18$ may have any one of the eleven values

360, 180, 120, 90, 72, 60, 45, 40, 36, 30, 24.

The 12th value 20 is inadmissible, since $n > 2$.

$$\therefore n = 342,\ 162,\ 102,\ 72,\ 54,\ 42,\ 27,\ 22,\ 18,\ 12,\ 6,$$
$$m = \ 19,\ \ 18,\ \ 17,\ 16,\ 15,\ 14,\ 12,\ 11,\ 10,\ \ 8,\ 5.$$

5. Let $CBEF$ be any quadrilateral. Produce $BE$ and $CF$ through $E$ and $F$ to meet in $A$, and $CB$, $FE$ through $B$ and $E$ to meet in $D$.

We will first prove that the circum-circles of the four triangles thus formed pass through a common point.

Let the circles round $AEF$ and $BED$ intersect in $K$. Join $FK, KD, KE$.

Then the angle $FKD = FKE + EKD = FAE + EBC = \pi - ACB$.

$\therefore$ the circle round $CDF$ passes through $K$.

Join $AK, KB$. Then $AKB = BKE + EKA = BDE + CFE = \pi - ACB$.

$\therefore$ the circle round $ABC$ also passes through $K$.

Let $O, P, Q, R$ be the centres of the circles round $BED$, $AEF$, $ABC$, $CDF$.

Then since the line joining the centres of two intersecting circles is perpendicular to their common chord $\therefore$ $RO$ and $OP$ are perpendicular respectively to $KD$ and $KE$. $\therefore$ the angle $POR = EKD = ABC$. Similarly $RQ$ and $PQ$ are perpendicular to $CK$ and $AK$, $\therefore$ $PQR$ = sup. of $AKC = \pi - ABC$. $\therefore$ $POR + PQR = 2$ right angles. $\therefore$ a circle will go round $OPQR$.

It can be shewn that the point $K$ also lies on this circle.

For $$OKE = \frac{\pi}{2} - POK = \frac{\pi}{2} - EDK = \frac{\pi}{2} - EBK,$$

and $$PKE = \frac{\pi}{2} - OPK = \frac{\pi}{2} - EAK,$$

$$\therefore OKP = \pi - (EBK + EAK) = AKB = \pi - ACB = FKD$$
$$= \pi - ORP, \text{ since } RO \text{ and } RP \text{ are perpendicular to } KD \text{ and } KF.$$
$$\therefore OKP + ORP = 2 \text{ right angles.}$$
$$\therefore K \text{ lies on the circle round } POR.$$

NOTE.—By the aid of XXVII. No. 5 we can shew that the ortho-centres of the triangles formed by four straight lines in a plane lie on a straight line, viz. the directrix of the parabola which can be described touching the four given straight lines.

6. Let $(x'y')$ be the point from which the tangents are drawn.

Let $px + qy = 1$ (1) be the equation to the tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then $a^2p^2 + b^2q^2 = 1$. . . . . (2)

The equation to a straight line through $(x'y')$ at right angles to (1) is

$$q(x - x') - p(y - y') = 0,$$

or $$qx - py = qx' - py'.$$

If this touches $\frac{x^2}{a^2} + \frac{y^2}{\beta^2} = 1$, we have

$$a^2q^2 + \beta^2p^2 = (qx' - py')^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and since $(x'y')$ is on (1) $\quad \therefore px' + qy' = 1$. . . . . (4)

We have to eliminate $p$ and $q$ between (2), (3) and (4).

From (2) and (4) $a^2p^2 + b^2q^2 = 1 = p^2x'^2 + q^2y'^2 + 2pqx'y'$.

From (3) $\quad \beta^2p^2 + a^2q^2 \quad = q^2x'^2 + p^2y'^2 - 2pqx'y'$.

$$\therefore p^2(a^2 + \beta^2) + q^2(a^2 + b^2) = (p^2 + q^2)(x'^2 + y'^2),$$

$$\therefore p^2(a^2 + \beta^2 - x'^2 - y'^2) = q^2(x'^2 + y'^2 - a^2 - b^2). \quad . \quad (5)$$

From (3) $\quad 4p^2q^2x'^2y'^2 = \left\{p^2(\beta^2 - y'^2) + q^2(a^2 - x'^2)\right\}^2$. . . . (6)

$\therefore$ substituting in (6) the value of the ratio $p^2 : q^2$ from (5), the equation of the locus becomes

$$4x^2y^2(a^2 + \beta^2 - x^2 - y^2)(x^2 + y^2 - a^2 - b^2)$$
$$= \left\{(x^2 - a^2)(x^2 - a^2) - (y^2 - b^2)(y^2 - \beta^2)\right\}^2. \quad . \quad (7)$$

If $P$ be any point on the curve, $O$ the origin, and $a^2 + \beta^2 > a^2 + b^2$ then if $OP^2 (= x^2 + y^2)$ be $> a^2 + \beta^2$, the expression on the left is negative, and $\therefore$ the expression on the right, which is a perfect square, is negative, which is impossible. $\therefore$ no part of the curve lies outside the larger circle. Similarly it can be shewn that no part lies within the smaller circle.

If we put $x^2 + y^2 = a^2 + \beta^2$, (7) becomes

$$0 = \left\{(x^2 - a^2)(x^2 + y^2 - a^2 - b^2)\right\}^2.$$

Now since $a^2 + \beta^2 > a^2 + b^2$, we cannot have $x^2 + y^2 - a^2 - b^2 = 0$,

$$\therefore x^2 - a^2 = 0 = y^2 - \beta^2.$$

$\therefore$ the curve meets the larger circle in the real points given by $x^2 = a^2$, $y^2 = \beta^2$, and it has been proved that no part of the curve lies outside this circle, $\therefore$ the curve touches the larger circle at these points. Similarly it can be shewn that the curve touches the smaller circle in the points given by $x^2 = a^2$, $y^2 = b^2$.

7. Let $O$ be the point of projection, $A$ the position of any one of the particles at the time $t$. Refer the position of $A$ to three rectangular axes through $O$ as origin, the plane of $xy$ being horizontal, and the axis of $z$ vertical. Let $u_1$ be the initial velocity of $A$, $a_1$ the angle of projection, and let $OM$, the intersection of the plane of $A$'s projection with the plane of $xy$ make an angle $A$ with $Ox$. Draw $MN$ perpendicular to $Ox$. Then if $x_1, y_1, z_1$ be the coordinates of $A$,

$$x_1 = ON = OM \cos A = u_1 \cos a_1 \cos A \,.\, t,$$

$$y_1 = MN = OM \sin A = u_1 \cos a_1 \sin A \,.\, t,$$

$$z_1 = AM = u_1 \sin a_1 \,.\, t - \tfrac{1}{2} g t^2,$$

with similar expressions for the coordinates of $B$ and $C$.

Now $$BC^2 = (x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2,$$

and since the quantities $u_1$, $a_1$, $A$, &c. are constant

$$\therefore BC^2 \propto t^2 = a^2 t^2 \text{ suppose.}$$

$\therefore$ the area of the triangle $ABC$

$$= t^2 \sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4} \propto t^2.$$

Next suppose that $A$ and $B$ are projected in the same vertical plane from the point $O$. Then if the plane of the triangle passes through $O$, it is obvious that $AB$ passes through $O$. Draw $AM$ and $BN$ perpendiculars on the horizontal line in the plane through $O$.

Let $AOM = \theta$. Then $OM = u \cos a \,.\, t$; $\quad AM = u \sin a \,.\, t - \tfrac{1}{2} g t^2$.

$$\therefore \tan \theta = \frac{AM}{OM} = \frac{u \sin a \,.\, t - \tfrac{1}{2} g t^2}{u \cos a \,.\, t},$$

$$\therefore t = (\tan a - \tan \theta) \frac{2u \cos a}{g}.$$

Similarly $$t = (\tan \beta - \tan \theta) \frac{2v \cos \beta}{g},$$

$$\therefore \tan \theta \,(v \cos \beta - u \cos a) = v \sin \beta - u \sin a,$$

$$\therefore t = \frac{2u \cos a}{g} \left\{ \tan a - \frac{u \sin a - v \sin \beta}{u \cos a - v \cos \beta} \right\}$$

$$= \frac{2}{g} \cdot \frac{uv \sin (\beta - a)}{u \cos a - v \cos \beta}.$$

## PAPER LXXXVIII.

1. (1) $\sqrt{x+a} + \sqrt{x+2a} = \sqrt{x+6a} - \sqrt{x+3a}$,

$\therefore x+a+x+2a+2\sqrt{x^2+3ax+2a^2} = x+6a+x+3a-2\sqrt{x^2+9ax+18a^2}$,

$\therefore \sqrt{x^2+3ax+2a^2} = 3a - \sqrt{x^2+9ax+18a^2}$,

$\therefore x^2+3ax+2a^2 = 9a^2+x^2+9ax+18a^2 - 6a\sqrt{x^2+9ax+18a^2}$,

$\therefore 36(x^2+9ax+18a^2) = (25a+6x)^2 = 625a^2+300ax+36x^2$,

$\therefore 23a^2 = -24ax$, $\therefore x = -\frac{23}{24}a$.

(2) $\sqrt{(x^2+a^2)(x^2+b^2)} - (nb^2+x^2) = x\left\{\sqrt{x^2+b^2} - \sqrt{x^2+a^2}\right\}$,

$\therefore$ squaring and transposing we get

$$2nb^2\sqrt{(x^2+a^2)(x^2+b^2)} = 2nb^2x^4 + n^2b^4 + a^2b^2.$$

Square again, and transpose.

$$\therefore 4nx^2b^4\left\{n(a^2+b^2) - n^2b^2 - a^2\right\} = b^4(n^4b^4 - 2n^2a^2b^2 + a^4),$$

$$\therefore x^2 = \frac{(n^2b^2-a^2)^2}{4n(n-1)(a^2-nb^2)}.$$

(3) $x^2+y^2+z^2 = a^2+2x(y+z)-x^2$,

$\therefore (x-y)^2+(x-z)^2 = a^2$ . . . . . (1)

Similarly $(y-x)^2+(y-z)^2 = b^2$ . . . . . (2)

$(z-x)^2+(z-y)^2 = c^2$ . . . . . (3)

$\therefore$ by addition

$$(x-y)^2+(y-z)^2+(z-x)^2 = \frac{a^2+b^2+c^2}{2},$$

and $(y-z)^2+(z-x)^2 = c^2$, $\therefore (x-y)^2 = \frac{a^2+b^2-c^2}{2}$.

Similarly $(y-z)^2 = \frac{b^2+c^2-a^2}{2}$, $(z-x)^2 = \frac{c^2+a^2-b^2}{2}$.

From these the values of $x$, $y$, and $z$ cannot be found, but expressing the fact that $x-y+y-z+z-x = 0$, we obtain a relation between $a$, $b$ and $c$.

2. $$y \sin 3\theta + x \cos 3\theta = (x^2 + y^2) \cos^3 \theta \quad . \quad . \quad . \quad (1)$$
$$y \cos 3\theta - x \sin 3\theta = (x^2 + y^2) \sin^3 \theta \quad . \quad . \quad . \quad (2)$$

Square and add.

$$\therefore x^2 + y^2 = (x^2 + y^2)^2 (\cos^6 \theta + \sin^6 \theta)$$
$$\therefore 1 = (x^2 + y^2)(1 - 3 \sin^2 \theta \cos^2 \theta) \quad . \quad . \quad . \quad (3).$$

Multiply (1) by $\cos 3\theta$, (2) by $\sin 3\theta$, and subtract.

$$\therefore x = (x^2 + y^2) \{\cos^3 \theta \cos 3\theta - \sin^3 \theta \sin 3\theta\}$$
$$= (x^2 + y^2) \{4 (\cos^6 \theta + \sin^6 \theta) - 3(\cos^4 \theta + \sin^4 \theta)\}$$
$$= (x^2 + y^2)(1 - 6 \sin^2 \theta \cos^2 \theta)$$
$$\therefore x^2 + y^2 + x = (x^2 + y^2)(2 - 6 \sin^2 \cos^2 \theta)$$
$$= 2, \text{ by (3).}$$

3. If $r_1, r_2, r_3$ be the radii of the escribed circles,

$$r_1 + r_2 + r_3 = S \left\{ \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \right\}$$
$$= \frac{S}{(s-a)(s-b)(s-c)} \left\{ bc + ca + ab - s^2 \right\}$$
$$r_2 + r_3 - r_1 = S \left\{ \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s-a} \right\}$$
$$= \frac{S}{(s-a)(s-b)(s-c)} \left\{ s^2 - a^2 - bc \right\},$$
$$r_2 r_3 = \frac{S^2}{(s-b)(s-c)},$$
$$\therefore \cos^2 \frac{a}{2} = \frac{(r_2 + r_3 + r_1)(r_2 + r_3 - r_1)}{4 r_2 r_3}$$
$$= \frac{(bc + ca + ab - s^2)(s^2 - a^2 - bc)}{4(s-a)^2 (s-b)(s-c)},$$
$$\therefore a(s-a) \cos^2 \frac{a}{2} = \frac{(bc + ca + ab - s^2)(as^2 - a^3 - abc)}{4(s-a)(s-b)(s-c)},$$

$\therefore$ the given expression on the left

$$= \frac{(bc + ca + ab - s^2)\{s^2(a + b + c) - a^3 - b^3 - c^3 - 3abc\}}{4(s-a)(s-b)(s-c)}$$

Now if we substitute $\frac{a+b+c}{2}$ for $s$, and multiply out, we find

$$s^2(a+b+c) - a^3 - b^3 - c^3 - 3abc$$

$$= \tfrac{3}{4}(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 - a^3 - b^3 - c^3 - 2abc),$$

and $4(s-a)(s-b)(s-c)$ in a similar manner can be shewn equal to $\frac{1}{2}$ the same expression,

$\therefore$ the given expression $= \frac{3}{2}(bc + ca + ab - s^2)$.

4. Let $F$ be the middle point of $BD$. Since $AEF$ cuts the sides of the triangle $BCD$, $\therefore DF \,.\, BA \,.\, CE = FB \,.\, CA \,.\, ED$, and $DF = FB$;

$\therefore BA \,.\, CE = CA \,.\, ED$. $\therefore CE : ED :: AC : AB :: BC : CA$.

5. Let $AL$ be the fixed line, $S$ the fixed point. Draw $SA$ perpendicular to $AL$, and produce $A$ to $X$ so that $SA = AX$. Through $X$ draw $XK$ parallel to $AL$. Then $XK$ is a fixed straight line. Let $Y$ be any position of the angular point of the right angle, so that $SY$, $YP$ are the positions of the arms. Produce $SY$ to meet $XK$ in $K$, and draw through $K$ a straight line at right angles to $XK$, meeting $YP$ in $P$. Join $SP$. Then $SY : YK :: SA : AX$. $\therefore SY = YK$. And $YP$ is common to the two triangles $SPY$, $KPY$, and at right angles to $SK$. $\therefore$ the angle $SPY = KPY$, and $SP = PK$. $\therefore$ the straight line $YP$ meets the parabola at $P$. Suppose that it also meets it at $P'$. Join $SP'$ and draw $P'K'$ perpendicular to $XK$, and produce $PY$ to meet $XK$ in $T$. Join $ST$. Then in the triangles $SPT$, $KPT$, $SP$, $PT = KP$, $PT$, and the angle $SPT = KPT$, $\therefore$ the angle $PST = PKT =$ a right angle.

Now $SP = PK$ and $SP' = P'K'$,

$\therefore SP : SP' :: PK : P'K' :: TP : TP'$.

$\therefore TS$ bisects the angle between $PS$ and $P'S$ produced, and is $\therefore$ at right angles to the line bisecting the angle $PSP'$. But $TS$ has been shewn to be perpendicular to $SP$. $\therefore P'$ must he indefinitely near to $P$. $\therefore$ the straight line $YP$ passes through two consecutive points on the curve, and is $\therefore$ a tangent.

6. Let $x^2 - y^2 = a^2$ be the equation of an equilateral hyperbola. If $x'y'$ he any point on the curve, the equation of the normal at $x'y'$ is

$$xy' + yx' = 2x'y'.$$

This expresses the relation between $(x'y')$ and the coordinates of any point $(xy)$ on the normal. If the point on the normal be given, $(h, k)$ suppose, to find $x'y'$ we have

$$2x'y' = hy' + kx' \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

This represents a curve passing through the points where the normals from $(h, k)$ meet the given hyperbola. Now $(A)$ represents a hyperbola whose asymptotes are parallel to the axes. $\therefore$ it is rectangular. Now by Bes. *Con.* Art. 138, the four points in which two rectangular hyperbolas intersect are such that any one of them is the orthocentre of the triangle formed by the other three.

7. Tripos 1875. Tuesday morning. No. 5.

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## Paper LXXXIX.

1. (1) Square and transpose.

$$\therefore \sqrt{\frac{a+x}{a-x}} + \sqrt{\frac{a-x}{a+x}} = \sqrt{b} - 2$$

$$\therefore \frac{a+x}{a-x} + \frac{a-x}{a+x} + 2 = b - 4\sqrt{b} + 4$$

$$\therefore \frac{a^2+x^2}{a^2-x^2} = \frac{b - 4\sqrt{b} + 2}{2}$$

$$\therefore \frac{x^2}{a^2} = \frac{b - 4\sqrt{b}}{b - 4\sqrt{b} + 4}$$

$$\therefore x = \pm \frac{a}{\sqrt{b} - 2} \cdot \sqrt{b - 4\sqrt{b}}.$$

(2) For $x + y$ write $z$, and add 1 to both sides of the first equation.

$$\left.\begin{aligned} \therefore x^2z^2 + (z+1)^2 &= 136 \\ 2xz(z+1) &= 120 \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore zx + z + 1 &= \pm 16 \\ zx - (z+1) &= \pm 4 \end{aligned}\right\}$$

$$\therefore z + 1 = \pm 10, \pm 6; \quad \therefore z = 9, -11, 5, -7,$$

and $zx = z + 1 \pm 4$.

$$\left.\begin{aligned} \therefore \text{(1) } 9x &= 10 \pm 4 = 14 \text{ or } 6 \\ \text{(2) } -11x &= -10 \pm 4 = -6 \text{ or } -14 \\ \text{(3) } 5x &= 6 \pm 4 = 10 \text{ or } 2 \\ \text{(4) } -7x &= -6 \pm 4 = -2 \text{ or } -10 \end{aligned}\right\}$$

The corresponding values of $y$ are given by $y = z - x$.

(3) Subtract the second equation from the third.

$$\therefore\ y - z + x(y - z) = 0,\quad \therefore\ (x + 1)(y - z) = 0.$$

$x + 1 = 0$, though true, gives an indeterminate equation to determine $y$ and $z$.

If $y - z = 0$, for $z$ write $y$ in (1),

$\therefore\ y^2 + 2y = 3,\quad \therefore\ y = 1$ or $-3$; $\therefore\ z = 1$ or $-3$. If we substitute these values in the given equations, we get the single value $x = -1$.

2. $$A + B + C = \pi,\quad \therefore\ 2A + 2B + 2C = 2\pi,$$

$$\therefore\ (\pi - 2A) + (\pi - 2B) + (\pi - 2C) = \pi,$$

and $$\sin(\pi - 2A) = \sin 2A;\ \cos(\pi - 2A) = -\cos 2A.$$

$\therefore$ all formulæ which hold for $A$, $B$, $C$ hold for $2A$, $2B$, $2C$ by changing the signs of the cosines.

Again $5A + 5B + 5C = 5\pi$, $\therefore\ (2\pi - 5A) + (2\pi - 5B) + (2\pi - 5C) = \pi$, and $\sin(2\pi - 5A) = -\sin 5A$; $\cos(2\pi - 5A) = \cos 5A$.

$\therefore$ all formulæ which hold for $A$, $B$, $C$ hold for $5A$, $5B$, $5C$ by changing the signs of the sines.

Now by Todh. *Trig.* Art 114,

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C,$$

$$\therefore\ \sin 10A + \sin 10B + \sin 10C = 4 \sin 5A \sin 5B \sin 5C,$$

and by the same Article, when $A + B + C = \frac{\pi}{2}$

$$1 = \tan B \tan C + \tan C \tan A + \tan A \tan B,$$

$$\therefore\ \cot A + \cot B + \cot C = \cot A \cot B \cot C.$$

Now $$\frac{5\pi + A}{2^5} + \frac{5\pi + B}{2^5} + \frac{5\pi + C}{2^5} = \frac{16\pi}{32} = \frac{\pi}{2},$$

$$\therefore\ \cot\frac{5\pi + A}{2^5} + \cot\frac{5\pi + B}{2^5} + \cot\frac{5\pi + C}{2^5}$$

$$= \cot\frac{5\pi + A}{2^5} \cot\frac{5\pi + B}{2^5} \cot\frac{5\pi + C}{2^5}.$$

3. Let $ABC$ be the equilateral triangle, $B$ being on the line which lies between the other two. Let $a$ be the distance between the lines on which $A$ and $B$ lie, $b$ the distance between the lines through $B$ and $C$. Then $a + b = c$. Let $BA$, $BC$ make angles $\phi$, $\theta$ with the line through $B$. Then $\theta + \phi = 60°$.

Then $c^2 = a^2 + b^2 + 2ab$, $\therefore\ a^2 + b^2 + c^2 = 2(a^2 + ab + b^2)$,

$a = AB \sin\phi$, $b = BC \sin\theta = AB \sin(60° - \phi)$

$$\therefore\ a^2 + b^2 + ab = \frac{AB^2}{2}\{2\sin^2\phi + 2\sin^2(60° - \phi) + 2\sin\phi\sin(60° - \phi)\}$$

$$= \frac{AB^2}{2}\{1 - \cos 2\phi + 1 - \cos(120 - 2\phi) + \cos(2\phi - 60°) - \cos 60°\}$$

$$= \frac{AB^2}{2}\{\tfrac{3}{2} - \cos 2\phi + 2\sin 30° \sin(90° - 2\phi)\}$$

$$= \tfrac{3}{4} AB^2.$$

Now area of $ABC = \frac{1}{2} AB \cdot AB \sin 60° = \frac{\sqrt{3}}{4} AB^2$

$$= \frac{1}{\sqrt{3}}(a^2 + b^2 + ab) = \frac{1}{2\sqrt{3}}(a^2 + b^2 + c^2).$$

NOTE.—To construct this triangle see Catalan, *Geom. El.* p. 66.

4. Let $D$ be the middle point of $RQ$.

$RCQ = 2RPQ$, $\therefore\ DCQ = RPQ$. $\therefore$ a circle will go round $QCNP$.

$\therefore\ RNC = MNP = CQM$; and $RCN = QCM$, $\therefore$ the triangles $RCN$, $QCM$ are equiangular, and $\therefore$ similar.

5. Let the parabolas have a common tangent at $C$. Through $C$ draw any two chords $CpP$, $CqQ$. Join $PQ$, $pq$. Then since all parabolas are similar curves $\therefore\ CP : Cp :: CQ : Cq$. $\therefore\ PQ$ and $pq$ are parallel. If one of the chords be turned about $C$ until it becomes indefinitely near to the other, $PQ$ and $pq$ become the directions of the tangents at $P$ and $p$. $\therefore$ the tangents at the extremities of any chord through $C$ are parallel. By the converse of this we obtain the required theorem.

6. By symmetry the circle touches the axis minor at the centre of the ellipse. $\therefore$ taking this point as origin, its equation is of the form

$$x^2 + y^2 + Ax = 0 \quad . \quad . \quad . \quad . \quad (1)$$

We can write the equation of the ellipse in the form

$$y^2 + \frac{b^2}{a^2} x^2 - b^2 = 0 \quad . \quad . \quad . \quad (2)$$

Since the circle and ellipse have double contact, the equation of the circle is

$$y^2 + \frac{b^2}{a^2}x^2 - b^2 = (mx + n)^2,$$

or
$$y^2 + x^2\left(\frac{b^2}{a^2} - m^2\right) - 2mnx - (b^2 + n^2) = 0 \quad . \quad . \quad . \quad (3)$$

Since (1) and (3) represent the same circle,

$$\therefore \frac{b^2}{a^2} - m^2 = 1\,;\ n^2 = -b^2,\ \ \therefore m^2n^2 = \frac{b^2}{a^2}(a^2 - b^2),$$

$$\therefore \text{(1) becomes } x^2 + y^2 \pm \frac{2b}{a}\sqrt{a^2 - b^2}\,x = 0,$$

and the radius of this circle is evidently $\frac{b}{a}\sqrt{a^2 - b^2}$.

7. If $v$ denote the velocity in feet per second, $v = 88$. If $r$ be the radius of the curve, $g$ the vertical acceleration, the horizontal acceleration $= \frac{v^2}{r}$, and the resultant acceleration $= \sqrt{g^2 + \frac{v^4}{r^2}}$.

If $N$ denote the number of oscillations in any time, $l$ the length of the pendulum, $N \propto \sqrt{\frac{g}{l}}$.

$$\therefore \left(\frac{\text{new number of oscillations}}{\text{former} \quad ,, \quad ,,}\right)^2 = \frac{\text{new resultant acceleration}}{\text{former} \quad ,, \quad ,,}$$

$$= \frac{\sqrt{g^2 + \frac{v^4}{r^2}}}{g} = \sqrt{1 + \frac{v^4}{r^2g^2}},$$

$$\therefore \left(\frac{121}{120}\right)^4 = 1 + \frac{v^4}{g^2r^2},$$

$$\therefore \frac{v^4}{g^2r^2} = (1 + \tfrac{1}{120})^4 - 1 = \tfrac{1}{30} \text{ nearly},$$

$$\therefore r = \frac{v^2}{g}\sqrt{30} = \frac{88 \times 88}{32}\sqrt{30} \text{ feet} = \tfrac{1}{4} \text{ mile nearly.}$$

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## Paper XC.

1. $S = 3 + 2 + 29 + 36 + 137 + 122 + 429 + 200 + \ldots$

$S_1 = \quad -1 + 27 + 7 + 101 - 15 + 307 - 229$

$S_2 = \quad 28 - 20 + 94 - 116 + 322 - 536$

$S_3 = \quad -48 + 114 - 210 + 438 - 858$

$S_4 = \quad 162 - 324 + 648 - 1296$

Here the fourth difference series is a series in G.P. the common ratio being $-2$. $\therefore$ the $n^{\text{th}}$ term is of the form

$$A + Bn + Cn(n+1) + Dn(n+1)(n+2) + E(-2)^{n-1}.$$

As in No. II. p. 19, determine $A$, $B$, $C$, $D$, $E$ by giving to $n$ in succession the values 1, 2, 3, 4, 5. We then find

$$A = 0,\ B = 3,\ C = -4,\ D = 1,\ E = 2.$$

$\therefore$ the $n^{\text{th}}$ term is $3n - 4n(n+1) + n(n+1)(n+2) + 2(-2)^{n-1}$,

$\therefore$ the sum of $n$ terms of the series

$$= \tfrac{3}{2}n(n+1) - \tfrac{4}{3}n(n+1)(n+2) + \tfrac{1}{4}n(n+1)(n+2)(n+3) + \tfrac{2}{3}\{1 - (-2)^n\}$$

$$= \tfrac{1}{12}n(n+1)^2(2n-4) + \tfrac{2}{3}\{1 - (-2)^n\}.$$

(2) Consider the series formed by the numerators, viz.,

$$3 + 8 + 20 + 50 + 128 + 338 + 920 + \ldots$$

It will be found that this comes under the form considered in No. IV., p. 21, $\therefore$ introducing $x$, &c.

$S = 3 + 8x + 20x^2 + 50x^3 + 128x^4 + 338x^5 + 920x^6 + \ldots$

$pxS = \quad 3px + 8px^2 + 20px^3 + 50px^4 + 128px^5 + 338px^6 + \ldots$

$qx^2S = \quad 3qx^2 + 8qx^3 + 20qx^4 + 50qx^5 + 128qx^6 + \ldots$

$rx^3S = \quad 3rx^3 + 8rx^4 + 20rx^5 + 50rx^6 + \ldots$

Add, and assume that the coefficients of $x^3$, $x^4$, $x^5$ vanish.

$$\left.\begin{aligned} \therefore\ 3r + 8q + 20p + 50 &= 0 \\ 8r + 20q + 50p + 128 &= 0 \\ 20r + 50q + 128p + 338 &= 0 \end{aligned}\right\}$$

From these equations we get $p = -6$, $q = 11$, $r = -6$.

By trial we find that these values of $p$, $q$, $r$ make the coefficient of $x^6$ vanish.

$$\therefore S = \frac{3 + x(8 + 3p) + x^2(20 + 8p + 3q)}{1 - 6x + 11x^2 - 6x^3} = \frac{3 - 10x + 5x^2}{(1 - x)(1 - 2x)(1 - 3x)}$$

$$= \frac{1}{1 - 3x} - \frac{3}{1 - 2x} - \frac{1}{1 - x} \text{ by partial fractions.}$$

$\therefore$ the $n^{th}$ term is $3^{n-1} - 3 \cdot 2^{n-1} - 1$.

$\therefore$ the $n^{th}$ term of the given series

$$= \frac{3^{n-1} - 3 \cdot 2^{n-1} - 1}{6^{n-1}} = \frac{1}{2^{n-1}} - \frac{3}{3^{n-1}} - \frac{1}{6^{n-1}},$$

$\therefore$ the sum of $n$ terms $= \frac{2^n - 1}{2^{n-1}} - \frac{3^n - 1}{2 \cdot 3^{n-2}} - \frac{6^n - 1}{5 \cdot 6^{n-1}}$.

(3) This can be treated like No. I. p. 19.

The $n^{th}$ term is $\frac{n}{1 \cdot 3 \cdot 5 \ldots (2n+1)}$. This can be written

$$\tfrac{1}{2} \frac{2n}{1 \cdot 3 \cdot 5 \ldots (2n+1)} \left\{ \frac{(2n+1) - 1}{2n} \right\}$$

$$= -\tfrac{1}{2} \left\{ \frac{1}{1 \cdot 3 \cdot 5 \ldots (2n+1)} - \frac{1}{1 \cdot 3 \cdot 5 \ldots (2n-1)} \right\},$$

and the first term is $-\tfrac{1}{2} \left\{ \frac{1}{1 \cdot 3} - \frac{1}{1} \right\}$

$\therefore$ the sum of $n$ terms $= \tfrac{1}{2} \left\{ 1 - \frac{1}{1 \cdot 3 \cdot 5 \ldots (2n+1)} \right\}$.

(4) Employ the method given in Arts. 7 and 8, pp. 5 and 6.

The $n^{th}$ term of the given series is $n(n+1)^2 a_n$

$$= (n^3 + 2n^2 + n)a_n = n^3 a_n + 2n^2 a_n + n a_n,$$

$$\therefore Sn = n^2(n+3)2^{n-3} + n(n+1)2^{n-1} + n2^{n-1}$$

$$= n2^{n-3}(n^2 + 7n + 8).$$

2. In the formula $\cos a = 1 - \frac{a^2}{2!} + \frac{a^4}{4!} \ldots$

for $a$ write $2x$, $4x$, $6x$, &c., and neglect powers of $x$ higher than the

fourth. The numerator on reduction becomes $\frac{20160}{4!}x^4$, and the denominator becomes $\frac{192}{4!}x^4$. $\therefore$ the limiting value of the given fraction $= \frac{20160}{192} = 105$.

3. The angle $B_1A_1C_1 = \frac{\pi}{2} - A_1BA = B$,

$$B_1C_1 = B_1C + CC_1 = b\cot A + \frac{a}{\sin C} = \frac{b\sin C\cos A + a\sin A}{\sin A\sin C},$$

$\therefore$ if $R_1$ be the radius of the circle round $A_1B_1C_1$,

$$R_1 = \frac{B_1C_1}{2\sin A_1} = \frac{a}{2\sin A\sin B\sin C}\left\{\frac{\sin B\sin C\cos A}{\sin A} + \sin A\right\}$$

$$= \frac{R}{2\sin A\sin B\sin C}\left\{2\sin^2 A + 2\sin B\sin C\cos A\right\}.$$

Now $2\sin B\sin C\cos A = \sin B\left\{\sin(A+C) - \sin(A-C)\right\}$

$$= \sin^2 B - \sin(A+C)\sin(A-C)$$

$$= \sin^2 B - \sin^2 A + \sin^2 C.$$

$$\therefore R_1 = R\frac{\sin^2 A + \sin^2 B + \sin^2 C}{2\sin A\sin B\sin C}.$$

Similarly we can shew that

$$R_2 = R_1\frac{\sin^2 A + \sin^2 B + \sin^2 C}{2\sin A\sin B\sin C} = R\left(\frac{\sin^2 A + \sin^2 B + \sin^2 C}{2\sin A\sin B\sin C}\right)^2$$

$$\therefore R_n = R\left(\frac{\sin^2 A + \sin^2 B + \sin^2 C}{2\sin A\sin B\sin C}\right)^n.$$

4. Join $A'B$, and produce $DD'$ to meet $A'B$ in $E$. Produce $A'D'$ to meet $BD$ in $G$. Then since $ED'D$ is a transversal of $A'BG$,

$$\therefore BD.GD'.EA' = GD.A'D'.EB,$$

or

$$AC.B'B.EA' = AA'.CB'.BE,$$

$\therefore AEB'$ is a transversal of $A'BC$.

5. Let the equations to $(A)$ and $(B)$ be

$$y^2 = 4ax \quad (1); \qquad y^2 = -4ax \quad (2)$$

The pole of $(h, k)$ with respect to $B$ is

$$yk = -2a(x + h) \quad (3), \quad \therefore x = -\left(\frac{yk}{2a} + h\right).$$

Substitute for $x$ in (1),

$$\therefore y^2 - 2yk + 4ah = 0 \quad (4),$$

$\therefore$ if (3) is a tangent to $(A)$, (4) must have equal roots,

$$\therefore k^2 = 4ah, \quad \therefore (A) \text{ is the locus of } (h, k).$$

6. The equation to $PQ$ is $\frac{y - y'}{a^2y'} = \frac{x - x'}{b^2x'}$, where $(x'y')$ are the coordinates of $P$.

If $(\xi, \eta)$ be the coordinates of $Q$,

$$\frac{\eta - y'}{a^2y'} = \frac{\xi - x}{b^2x'} \quad (1).$$

Since $(\xi, \eta)$, $(x', y')$ are on the ellipse,

$$\therefore \frac{\eta^2}{b^2} + \frac{\xi^2}{a^2} = 1 = \frac{y'^2}{b^2} + \frac{x'^2}{a^2},$$

$$\therefore \frac{\eta^2 - y'^2}{b^2} = -\frac{\xi^2 - x'^2}{a^2} \quad (2).$$

Now (2) is satisfied by the coordinates of $P$, $P'$ and $Q$.

$\therefore$ dividing (2) by (1), the equation of $QP'$ is

$$\frac{a^2y'}{b^2}(y + y') = -\frac{b^2x'}{a^2}(x + x').$$

$$\therefore \text{if } y = 0, \; x = -\frac{a^2}{b^2x'}\left(\frac{b^2x'^2}{a^2} + \frac{a^2y'^2}{b^2}\right) = -CU; \text{ and } MG = \frac{b^2x'}{a^2},$$

$$\therefore CU \cdot MG = \frac{b^2x'}{a^2} + \frac{a^2y'}{b^2} = CD^2.$$

7. Let $e$ be the elasticity, $v$ the velocity, $a$ the angle of projection, $a$ the distance of the point of projection from the wall.

Then the time to the wall $= \dfrac{a}{v \cos a}$,

and the time back again $= \dfrac{a}{ev \cos a}$.

Also the whole time $= \dfrac{2v \sin a}{g}$,

$$\therefore v^2 . \sin 2a = ag(1 + e).$$

Now the right-hand side is constant, $\therefore$ $v$ is a min. when $\sin 2a$ is a max., *i.e.* when $a = \dfrac{\pi}{4}$.

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## Paper XCI.

1. Let $x, y$ denote the two quantities. Their H.M. is $\dfrac{2xy}{x+y}$.

The G.M. between $x$ and $\dfrac{2xy}{x+y}$ is $x\sqrt{\dfrac{2y}{x+y}}$. The G.M. between $\dfrac{2xy}{x+y}$ and $y$ is $y\sqrt{\dfrac{2x}{x+y}}$. $\therefore$ by question

$$\frac{4xy}{x+y} = x\sqrt{\frac{2y}{x+y}} + y\sqrt{\frac{2x}{x+y}}$$

$$2\sqrt{2}\sqrt{xy} = \sqrt{x+y}\,(\sqrt{x} + \sqrt{y}),$$

$$\therefore (x+y)^2 + 2\sqrt{xy}\,(x+y) = 8xy,$$

$$\therefore x + y = -\sqrt{xy} \pm \sqrt{9xy} = -4\sqrt{xy} \text{ or } 2\sqrt{xy}.$$

The value $x + y = 2\sqrt{xy}$ gives us $x = y$, which is inadmissible.

$$\therefore x + y = -4\sqrt{xy}, \quad \therefore \frac{x}{y} + 4\sqrt{\frac{x}{y}} + 1 = 0,$$

$$\therefore \sqrt{\frac{x}{y}} = -2 \pm \sqrt{3},$$

$$\therefore \frac{x}{y} = 7 \pm 4\sqrt{3}.$$

2. $(m + \frac{3}{2})^2 = m^2 + 3m + \frac{9}{4} > (m + 1)(m + 2)$

$(m + \frac{7}{2})^2 = m^2 + 7m + \frac{49}{4} > (m + 3)(m + 4)$

. . . . . .

$$\left(m + \frac{4n-1}{2}\right)^2 = m^2 + (4n-1)m + \left(\frac{4n-1}{2}\right)^2 > (m+2n)(m+2n-1)$$

$$\therefore (m + \tfrac{3}{2})^2 (m + \tfrac{7}{2})^2 \ldots \left(m + \frac{4n-1}{2}\right)^2 > (m + 2n) \ldots (m + 1),$$

$$\therefore (m + \tfrac{3}{2})(m + \tfrac{7}{2}) \ldots \left(m + \frac{4n-1}{2}\right) > \left\{\frac{(m+2n)!}{m!}\right\}^{\frac{1}{2}}$$

3. Let $y - 1, y, y + 1$ be the lengths of the sides.

Then $s = \frac{3y}{2}$, $s - a = \frac{y+2}{2}$, $s - b = \frac{y}{2}$, $s - c = \frac{y-2}{2}$,

$$\therefore S^2 = s(s-a)(s-b)(s-c) = \frac{y^2}{16} \cdot 3(y^2 - 4).$$

$\therefore 3(y^2 - 4)$ must be a perfect square, $= x^2$ suppose. Then $x$ is a multiple of 3. One solution is easily seen to be $x = 6, y = 4$.

Consider the equation $x^2 - 3y^2 = 1$.

$$\sqrt{3} = 1 + \frac{1}{1+} \frac{1}{2+} \quad \ldots\ldots$$

$\therefore$ Todh. *Alg.* Art. 639, one solution is $x = 2, y = 1$. By Art. 643 we obtain the other solutions thus.

$$(2 - \sqrt{3})^n (2 + \sqrt{3})^n = 1 = (x - \sqrt{3}y)(x + \sqrt{3}y).$$

Let $x - \sqrt{3}y = (2 - \sqrt{3})^n$ ; $x + \sqrt{3}y = (2 + \sqrt{3})^n$ ,

$$\therefore x = \tfrac{1}{2}\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}; \; y = \frac{1}{2\sqrt{3}}\{(2+\sqrt{3})^n - (2-\sqrt{3})^n\}.$$

By giving to $n$ in succession the values 1, 2, 3, 4 we get

$$\left.\begin{array}{l} x = 2, 7, 26, 99, \ldots \\ y = 1, 4, 15, 56, \ldots \end{array}\right\} A.$$

$\therefore$ Todh. *Alg.* Art. 645, we have $y = pn \pm qm$, where $p = 6$, $q = 4$, and $m$ and $n$ are corresponding values of $x$ and $y$ in $(A)$,

$$\therefore y = 4, 14, 52, 194, 724, \&c.$$

4. Let $O$ be the circum-centre, $P$ the ortho-centre. Bisect $OP$ in $N$. Then $N$ is a fixed point. Now the radius of the nine-points' circle $= \frac{1}{2}$ the radius of the circum-circle, and is $\therefore$ given, and its centre is at $N$. $\therefore$ the nine-points' circle, which passes through the middle points of the sides, is fixed.

5. If $H$ be the other focus, $Hq = SQ$, since $Qq$ is a diameter. Since $St$ bisects the angle $PSq$ $\therefore$ the angle $qSt = PSt = Stq$,

$$\therefore\ qt = Sq. \quad \text{Similarly } TQ = SQ.$$

$$\therefore\ TQ + tq = Sq + SQ = Sq + Hq = AA'.$$

6. Let $PSQ$ be the focal chord, $Q$ being nearer the vertex than $P$. Let the normals at $P$ and $Q$ meet the axis in $G$, $G'$; and let the tangents intersect in the directrix at $Z$. Let $T$ denote the tension of the string, which acts along the tangents at $P$ and $Q$.

Then
$$W_1 : T :: 1 : \sin SGP :: 1 : \cos SPZ,$$
$$W_2 : T :: 1 : \sin SG'Q :: 1 : \cos SQZ,$$
$$\text{and } SQZ = \frac{\pi}{2} - SPZ,$$
$$\therefore\ T = W_1 \cos SPZ = W_2 \sin SPZ,$$
$$\therefore\ \frac{W_1}{W_2} = \tan SPZ = \frac{SZ}{SP};\quad \frac{W_2}{W_1} = \tan SQZ = \frac{SZ}{SQ},$$
$$\therefore\ \frac{W_1}{W_2} + \frac{W_2}{W_1} = ZS \cdot \frac{SP + SQ}{SP \cdot SQ} = ZS \cdot \frac{PQ}{ZS^2} = \frac{PQ}{ZS}, \quad (1)$$
$$\text{and } PQ = \frac{4SP \cdot SQ}{\text{lat. rect.}} = \frac{4ZS^2}{\text{lat. rect.}}, \quad (2)$$

$\therefore$ from (1) and (2),

$$PQ^2 = ZS^2 \left(\frac{W_1}{W_2} + \frac{W_2}{W_1}\right)^2 = \tfrac{1}{4} \text{ lat. rect.} \left(\frac{W_1}{W_2} - \frac{W_2}{W_1}\right)^2 \cdot PQ.$$

7. Let $a$ be the inclination of the plane, $\phi$ the angle between this plane and the direction of projection, $v$ the velocity of projection, and $t$ the time of flight. Then resolving parallel and perpendicular to the given plane,

$$v \cos \phi - gt \sin a = 0\ ;\ vt \sin \phi - \tfrac{1}{2} gt^2 \cos a = 0,$$

$\therefore$ eliminating $t$, $\tan \phi = \frac{1}{2} \cot a$. $\therefore\ \phi = \tan^{-1} (\frac{1}{2} \cot a)$.

## PAPER XCII.

1. $(\sigma - a)^3 + (\sigma - b)^3 + (\sigma - c)^3 - 3(\sigma - a)(\sigma - b)(\sigma - c)$,

$= (\sigma - a + \sigma - b + \sigma - c)\{(\sigma - a)^2 + \ldots - (\sigma - b)(\sigma - c)\ldots\}$

$= \{3\sigma - (a+b+c)\}\{3\sigma^2 - 2\sigma(a+b+c) + a^2+b^2+c^2-3\sigma^2$
$\qquad +\sigma(b+c+c+a+a+b) - bc - ca - ab$

$= (a+b+c)(a^2+b^2+c^2 - bc - ca - ab)$

$= a^3 + b^3 + c^3 - 3abc.$

2. $r = \dfrac{S}{s}$, $r_1 = \dfrac{S}{s-a}$, &c. $\therefore\ rr_1r_2r_3 = S^2$,

$\therefore$ considering the last term on the right, we see that one of the radii must have a negative sign.

Now $-r_1r_2r_3 + rr_1r_2 + rr_2r_3 + rr_3r_1 = rr_1r_2r_3\left(-\dfrac{1}{r} + \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}\right)$

$= S^2 \cdot \dfrac{1}{S}(-s + s - a + s - b + s - c) = 0,$

$\therefore$ $r$ must have a negative sign.

$$-rr_1 - rr_2 - rr_3 + r_1r_2 + r_1r_3 + r_2r_3 = \frac{(a+b+c)^2}{2} - bc - ca - ab$$
$$= \frac{a^2+b^2+c^2}{2}$$

$$-r + r_1 + r_2 + r_3 = S\left(-\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right)$$
$$= \frac{1}{S}\{2s^3 - (a+b+c)s^2 + abc\} = \frac{abc}{S},$$

$\therefore$ $-r$, $r_1$, $r_2$, $r_3$ are the roots of the given equation.

3. $S_k = \dfrac{n^{k-1}}{k-1\,!} + \dfrac{(2n)^{k-2}}{k-2\,!} + \dfrac{(3n)^{k-3}}{k-3\,!} + \ldots$

$S_{k-1} = \dfrac{n^{k-2}}{k-2\,!} + \dfrac{(2n)^{k-3}}{k-3\,!} + \dfrac{(3n)^{k-4}}{k-4\,!} + \ldots$

. . . . . .

$$S_3 = \frac{n^2}{2!} + \frac{2n}{1} + 1,$$

$$S_2 = \frac{n}{1} + 1$$

$$S_1 = 1,$$

$$\therefore S_k + n S_{k-1} + \ldots + \frac{n^{k-2}}{k-2!} S_2 + \frac{n^{k-1}}{k-1!} S_1 + \frac{n^k}{k!}$$

$$= \frac{n^k}{k!} + \frac{n^{k-1}}{k-1!}\left\{1 + (k-1) + \frac{(k-1)(k-2)}{2!} + \ldots + (k-1) + 1\right\}$$

$$+ \frac{n^{k-2}}{k-2!}\left\{2^{k-2} + (k-2)\,2^{k-3} + \frac{(k-2)(k-3)}{2!}\,2^{k-4} + \ldots\right\}$$

$$+ \frac{n^{k-3}}{k-3!}\left\{3^{k-3} + (k-3)\,3^{k-4} + \ldots\ldots\ldots\right\}$$

$$+ \ldots\ldots$$

$$= \frac{n^k}{k!} + \frac{n^{k-1}}{k-1!}(1+1)^{k-1} + \frac{n^{k-2}}{k-2!}(2+1)^{k-2} + \ldots$$

$$= \frac{n^k}{k!} + \frac{(2n)^{k-1}}{k-1!} + \frac{(3n)^{k-2}}{k-2!} + \ldots$$

$$= S_{k+1}.$$

4. Since $APB$ is a right angle, $AB^2 = AP^2 + PB^2$.

Since $PQB$ is $>$ a right angle, $PB^2 > PQ^2 + QB^2$ by Euc. II. 12.

„ $QRB$ „ $QB^2 > QR^2 + RB^2$, &c.

$$\therefore AB^2 > AP^2 + PQ^2 + \ldots\ldots + KB^2.$$

5. The circle on $PQ$ as diameter touches the parabola at $P$ and passes through $K$. Bes. *Parab.* Art. 50. $\therefore$ $PKQ$ is a right angle.

6. After the piece has been cut out let $\bar{x}$ be the distance of the centre of gravity from the hollow end, and let $A$ be the area of a transverse section.

Then $\bar{x}\left(Aa - \frac{A}{n}\,.\,l\right) = Aa\,.\,\frac{a}{2} - A\,.\,\frac{l}{n}\,.\,\frac{l}{2}$,

$$\therefore \bar{x} = \frac{na^2 - l^2}{2(na - l)}.$$

$\therefore$ the distance moved $= \dfrac{na^2 - l^2}{2(na - l)} - \dfrac{a}{2} = \dfrac{l}{2} \cdot \dfrac{a - l}{na - l}$.

Put $\dfrac{l}{2} \cdot \dfrac{a - l}{na - l} = y$,

$$\therefore l^2 - al + 2nay - 2ly = 0,$$

$$\therefore l^2 - l(a + 2y) + 2nay = 0 \quad (1).$$

When $y$ is a max. (1) has equal roots.

$$\therefore l = \frac{a}{2} + y, \quad \therefore y = l - \frac{a}{2}.$$

$\therefore$ the point of support must be under the extremity of the bore.

7. Tripos 1878. Tuesday morning. No. 9.

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## Paper XCIII.

1. $$xz\left(\frac{1}{1 - x^2} \cdot \frac{1}{1 + z^2} - \frac{1}{1 + x^2} \cdot \frac{1}{1 - z^2}\right) = \frac{2xz(x^2 - z^2)}{(1 - x^4)(1 - z^4)},$$

$\therefore$ since $x, y, z$ are all unequal, the given equation reduces to

$$0 = xz(x^2 - z^2)(1 - y^4) + zy(z^2 - y^2)(1 - x^4) + yx(y^2 - x^2)(1 - z^4)$$

$$= (x^3z - xz^3 + y^3x - yx^3 + z^3y - zy^3) \quad (A)$$

$$- xyz(x^2y^3 - x^3y^2 + y^2z^3 - y^3z^2 + z^2x^3 - z^3x^2) \quad (B).$$

Consider the expression $A$. It vanishes when we put $x = y$, or $x = z$, or $y = z$, and is a symmetrical expression of the 4th degree.

$\therefore A \equiv K(x - y)(y - z)(z - x)(x + y + z)$, where $K$ is some numerical constant. Put $x = 2, y = 1, z = 0$, and we get $K = 1$.

Now consider $B$. It vanishes when we put $x = y$, or $x = z$, or $y = z$, and is a symmetrical expression of the 5th degree.

$$\therefore B = (x - y)(y - z)(z - x)\{C(x^2 + y^2 + z^2) + D(yz + zx + xy)\}$$

Equating coefficients of like terms, we get $C = 0$, $D = 1$.

$\therefore$ the given equation becomes

$$0 = (x-y)(y-z)(z-x)\{x+y+z-xyz(xy+yz+zx)\}$$
$$= x+y+z-xyz(xy+yz+zx).$$

2. $\sin^{-1}\frac{4}{5} + \cos^{-1}\frac{3}{5} = 2\sin^{-1}\frac{4}{5} = 2\tan^{-1}\frac{4}{3} = \tan^{-1}\frac{\frac{8}{3}}{1-\frac{16}{9}}$

$$= \tan^{-1}(-\tfrac{24}{7}),$$

$$\therefore\ -\tfrac{24}{7} = \tan\left\{\tan^{-1}(x+1) + \tan^{-1}\frac{1}{x-1}\right\}$$

$$= \frac{x+1+\frac{1}{x-1}}{1-\frac{x+1}{x-1}} = \frac{x^2}{-2},\quad \therefore\ x = \pm 4\sqrt{\tfrac{3}{7}}.$$

3.
$$\frac{\sin BAR}{\sin B} = \frac{BR}{AR} = \frac{2CR}{AR} = 2\frac{\sin CAR}{\sin C},$$

$$\frac{\sin CAQ}{\sin C} = \frac{CQ}{AQ} = \frac{2BQ}{AQ} = 2\frac{\sin BAQ}{\sin B},$$

$$\therefore\ \sin BAR \,.\, \sin CAQ = 4\sin CAR \sin BAQ.$$

4. Draw $BE$ perpendicular to $CD$. Then since the circles are equal, $\therefore$ the angle $BDC = BCD$. $\therefore$ the triangle $BCD$ is isosceles; $\therefore$ $E$ is the middle point of the base, and $E$ lies on the circle described on $AB$ as diameter.

5. Tripos 1878. Monday morning. No. 12.

6. Tripos 1875. 2nd Tuesday afternoon. No. 1.

7. Take the point of contact as origin, the common tangent, and the line joining the centres as axes. Let the coordinates of $P$ at any time be $(x_1, y_1)$. Then those of $P'$ at the same time are $(-x_1, -y_1)$. Similarly if $u$, $v$ be the component velocities of $P$ parallel to the axes, $-u$, $-v$ will be those of $P'$. $\therefore$ if $(x, y)$ be the relative coordinates of $P'$ with respect to $P$, $x = 2x_1$, $y = 2y_1$; and the component relative velocities of $P'$ with respect to $P$ are $2u$, $2v$. $\therefore$ $P'$ will appear to $P$ to be moving in a circle whose radius is equal to twice the radius of either circle.

## PAPER XCIV.

1. Tripos 1878. Wednesday morning. No. 5.

2. (1). Square and transpose.

$$\therefore\ \therefore\ 2 - (a^2 + b^2)(\tan^2\theta + \cot^2\theta) + 2a^2b^2$$
$$= 2\sqrt{\{1 - a^2(\tan^2\theta + \cot^2\theta) + a^4\}\{1 - b^2(\tan^2\theta + \cot^2\theta) + b^4\}}.$$

On squaring both sides and reducing, we obtain

$$(a^2 - b^2)^2(\tan^2\theta - \cot^2\theta)^2 = 0,$$

$$\therefore \tan\theta = \pm\cot\theta,\ \therefore \theta = n\pi \pm \frac{\pi}{4}.$$

(2). $$a\cos\theta + b\sin\theta = \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}b$$
$$= a\cos\frac{\pi}{4} + b\sin\frac{\pi}{4},$$

$$\therefore a\left(\cos\theta - \cos\frac{\pi}{4}\right) = -b\left(\sin\theta - \sin\frac{\pi}{4}\right),$$

$$\therefore 2a\sin\left(\frac{\pi}{8} - \frac{\theta}{2}\right)\sin\left(\frac{\pi}{8} + \frac{\theta}{2}\right) = 2b\sin\left(\frac{\pi}{8} - \frac{\theta}{2}\right)\cos\left(\frac{\pi}{8} + \frac{\theta}{2}\right),$$

$$\therefore \text{either } \frac{\theta}{2} - \frac{\pi}{8} = n\pi, \text{ or } \tan\left(\frac{\theta}{2} + \frac{\pi}{8}\right) = \frac{b}{a},$$

$$\therefore \frac{\theta}{2} + \frac{\pi}{8} = n\pi + \tan^{-1}\frac{b}{a}.$$

(3). $$4\sin^3\theta - \sin 3\theta = \frac{1}{\sqrt{2}} = 4\sin^3\frac{\pi}{4} - \sin\frac{3\pi}{4},$$

$$\therefore 4\left(\sin^3\theta - \sin^3\frac{\pi}{4}\right) = \sin 3\theta - \sin\frac{3\pi}{4}$$
$$= 3\sin\theta - 4\sin^3\theta - 3\sin\frac{\pi}{4} + 4\sin^3\frac{\pi}{4},$$

$$\therefore 8\left(\sin^3\theta - \sin^3\frac{\pi}{4}\right) = 3\left(\sin\theta - \sin\frac{\pi}{4}\right),$$

$\therefore$ either $\sin\theta - \sin\frac{\pi}{4} = 0$, $\therefore \sin\theta = \sin\frac{\pi}{4}$, $\therefore \theta = n\pi + (-1)^n \frac{\pi}{4}$,

or $$\sin^2\theta + \sin\theta \sin\frac{\pi}{4} + \sin^2\frac{\pi}{4} = \frac{3}{8},$$

$$\therefore \left(\sin\theta + \frac{1}{2\sqrt{2}}\right)^2 = 0, \quad \therefore \theta = n\pi + (-1)^n \left\{\sin^{-1}\left(-\frac{1}{2\sqrt{2}}\right)\right\}.$$

3. Express all the terms as sines and cosines, transpose all to the left-hand side, and bring to a common denominator. Then

$$\sec^2 B + \sec^2 C + 2\sec B \sec C \cos A - \sec B \sec C \sin A (\tan B + \tan C)$$

$$= \frac{1}{\cos^2 B \cos^2 C}\left\{\cos^2 C + \cos^2 B + 2\cos B \cos C \cos A - \sin^2 A\right\}$$

$$= \frac{1}{\cos^2 B \cos^2 C}\left\{\cos^2 B + 2\cos B \cos C \cos A + \cos(C+A)\cos(C-A)\right\}$$

$$= \frac{1}{\cos^2 B \cos^2 C}\cos B\left\{2\cos A \cos C - (\cos\overline{C+A} + \cos\overline{C-A})\right\}$$

$$= 0.$$

4. Consider the point $C$ as a point circle. Then $ED$ is the radical axis. $\therefore$ by Casey, p. 113. Cor. 1, $CP^2 = 2CO \,.\, MN$, where $O$ is the centre of the given circle, $M$ is the intersection of $DE$ and $CO$, and $N$ is the foot of the perpendicular from $P$ on $CO$.

5. Tripos 1875. Wednesday morning. No. 6.

6. Describe an ellipse passing through $O$, and confocal to the given ellipse, and let $O'$ be a point on the confocal near to $O$. The tangents from $O$ and $O'$ to the given ellipse intersect in $A$ and $B$. The normals at $O$ and $O'$ to the confocal bisect the angles $AOB$, $AO'B$. Now, with foci $A$ and $B$, describe a conic passing through $O$. Since its normal bisects the angle $AOB$, the two conics touch, and $\therefore$ ultimately they both pass through $O'$, and $\therefore$ their normals at $O$ coincide. $\therefore$ the two curves have the same curvature at $O$, and their central chords of curvature are equal.

Now central chord of curvature $= 2 \,.\, \dfrac{\text{product of focal distances}}{\text{distance from centre}}$.

$$\therefore \frac{OA \,.\, OB}{ON} = \frac{OS \,.\, OH}{OC}.$$

7. The ball is reflected each time at the same angle as it struck the circumference. ∴ during the time of falling only the vertical velocity is altered, and ∴ the intervals between the reflections are equal.

Let $u$ be the velocity of projection. Since the number of reflections depends only on the horizontal motion of the ball, we may suppose the ball to move in an equilateral polygon inscribed in a circle on a smooth horizontal table. The length of each side of the polygon is $2r \sin \frac{\pi}{n}$, ∴ if $t$ be the time of reaching the water,

$$ut = 2nr \sin \frac{\pi}{n},$$

But if $h$ is the depth, $t^2 = \frac{2h}{g}$.

$$\therefore \frac{2h}{g} = \frac{1}{u^2}\left(2nr \sin \frac{\pi}{n}\right)^2.$$

Now if $s$ be the space to which $u$ is due, $u^2 = 2gs$.

$$\therefore s = \frac{r^2}{h}\left(n \sin \frac{\pi}{n}\right)^2.$$

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## Paper XCV.

1. (1). The $n^{\text{th}}$ term $= n(n+2) = n(n+1) + n$.

∴ the sum of $n$ terms

$$= \frac{n(n+1)(n+2)}{3} + \frac{n(n+1)}{2} = \tfrac{1}{6}n(n+1)(2n+7).$$

(2). The $n^{\text{th}}$ term $= n(n+1)^2 = n(n+1)(n+2) - n(n+1)$.

∴ the sum of $n$ terms

$$= \frac{n(n+1)(n+2)(n+3)}{4} - \frac{n(n+1)(n+2)}{3} = \tfrac{1}{12}n(n+1)(n+2)(3n+5).$$

(3). The $n^{\text{th}}$ term $= \dfrac{n}{(2n-1)^2(2n+1)^2}$

$$= -\frac{1}{8}\left\{\frac{1}{(2n+1)^2} - \frac{1}{(2n-1)^2}\right\},$$

$$\text{1st term} = -\frac{1}{8}\left\{\frac{1}{3^2} - \frac{1}{1^2}\right\},$$

$$\therefore \text{ the sum of } n \text{ terms} = \frac{1}{8}\left\{1 - \frac{1}{(2n+1)^2}\right\}. \text{ See Errata.}$$

2. Let $x$, $xr$, $xr^2$ denote the number of degrees in the angles of the 1st triangle, $3x$, $3xR$, $3xR^2$ the number in the angles of the 2nd. Then we have

$$\left.\begin{aligned} x + xr + xr^2 &= 180 \\ 3x + 3xR + 3xR^2 &= 180 \\ xr^2 + 3xR^2 &= 240 \end{aligned}\right\}$$

$$\therefore \frac{1 + r + r^2}{1 + R + R^2} = 3\,; \quad \frac{1 + r + r^2}{r^2 + 3R^2} = \frac{3}{4},$$

$$\therefore 9R^2 + 3r^2 = 4r^2 + 4r + 4,$$

$$\therefore 9R^2 = r^2 + 4r + 4 = (r+2)^2, \quad \therefore 3R = r + 2,$$

$$\therefore 3(1 + r + r^2) = 9(1 + R + R^2) = 9 + 3r + 6 + r^2 + 4r + 4,$$

$$\therefore r^2 - 2r - 8 = 0, \quad \therefore (r-4)(r+2) = 0,$$

$$\therefore r = 4, \quad R = 2, \quad x = \tfrac{180}{21},$$

$\therefore$ angles of the 1st in degrees are $\frac{180}{21}, \frac{4 \times 180}{21}, \frac{16 \times 180}{21}$,

$\therefore$ ,, ,, 1st in circular measure are $\frac{\pi}{21}, \frac{4\pi}{21}, \frac{16\pi}{21}$,

,, ,, 2nd ,, ,, ,, $\frac{\pi}{7}, \frac{2\pi}{7}, \frac{4\pi}{7}$.

3. (1). The expression on the right

$$= \frac{3}{10}\left\{\log_e 10 + \frac{1}{2^7} + \frac{1}{2} \cdot \frac{3}{2^{14}} + \frac{1}{3} \cdot \frac{3^2}{2^{21}} + \ldots\right\}$$

$$= \frac{3}{10}\log_e 10 + \frac{1}{10}\left\{\frac{3}{2^7} + \frac{1}{2} \cdot \frac{3^2}{2^{14}} + \frac{1}{3} \cdot \frac{3^3}{2^{21}} + \ldots\right\}$$

$$= \frac{3}{10}\log_e 10 - \frac{1}{10}\log_e\left(1 - \frac{3}{2^7}\right)$$

$$= \frac{3}{10}\log_e 10 - \frac{1}{10}\log_e \frac{125}{2^7}$$

$$= \frac{3}{10}\log_e 10 - \frac{1}{10}\log_e \frac{10^3}{2^{10}}$$

$$= \frac{3}{10}\log_e 10 - \frac{3}{10}\log_e 10 + \log_e 2 = \log_e 2.$$

(2). $\left\{\log_e(1+x)\right\}^2 = \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n-1}}{n-1} + \ldots\right\}^2$.

In finding the coefficient of $x^n$ no term higher than $x^{n-1}$ will be required. If $n$ is even, the sign of each term will evidently be positive, and if $n$ is odd, the sign will be negative.

$$\therefore \text{ coef. of } x^n = (-1)^n\left\{\frac{1}{1(n-1)} + \frac{1}{2(n-2)} + \frac{1}{3(n-3)} + \ldots\right\}$$

$$= \frac{(-1)^n}{n}\left\{\frac{1}{1} + \frac{1}{n-1} + \frac{1}{2} + \frac{1}{n-2} + \frac{1}{3} + \frac{1}{n-3} + \ldots\right\}$$

$$= \frac{2(-1)^n}{n}\left\{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}\right\},$$

since each fraction will occur twice.

4. Produce $MD$ to meet $CN$ in $E'$.

Then $$MD : AE :: ND : NE :: DE' : EC.$$

But $AE = EC$, $\therefore MD = DE'$; and $BD = DC$, $\therefore BM$ and $CN$ are parallel.

5. Let the tangent at $P$ meet the conjugate diameters $CD$, $CD'$ in $M$ and $N$. Draw the ordinates $Pm$, $Pn$, $m$ being on $CD$.

Then $$CD^2 = CM \,.\, Cm = CM \,.\, Pn,$$

$$CD'^2 = CN \,.\, Cn = CN \,.\, Pm,$$

$$\therefore CD^2 \,.\, CD'^2 = CM \,.\, Pm \times CN \,.\, Pn,$$

$$\therefore CM \,.\, Pm \propto \frac{1}{CN \,.\, Pn}, \quad \therefore \Delta\, CPM \propto \frac{1}{\Delta\, CPN}.$$

6. Let the equations to the tangents be

$$y = m_1 x + c_1; \quad y = m_2 x + c_2.$$

The coordinates of the foci are $(x, y)$, $(-x, -y)$.

Then since $SY \cdot S'Y' = BC^2 = SZ \cdot S'Z'$,

$$\therefore \frac{(y - m_1x - c_1)(-y + m_1x - c_1)}{1 + m_1^2} = \frac{(y - m_2x - c_2)(-y + m_2x - c_2)}{1 + m_2^2},$$

$$\therefore \{c_1^2 - (y - m_1x)^2\}(1 + m_2^2) = \{c_2^2 - (y - m_2x)^2\}(1 + m_1^2),$$

$$\therefore c_1^2(1+m_2^2) - c_2^2(1+m_1^2) = (y^2 - x^2)(m_2^2 - m_1^2) + 2xy(m_2 - m_1)(1 - m_1m_2),$$

$$\therefore k^2 = y^2 - x^2 + 2xy \cdot \frac{1 - m_1m_2}{m_1 + m_2}$$

$$= y^2 - x^2 + 2xy \cot(\alpha + \beta).$$

7. Tripos 1878. Tuesday morning. No. 7.

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## Paper XCVI.

1. Tripos 1875. Monday morning. No. 2.

2. Tripos 1875. Wednesday morning. No. 7.

3. Let $a$ be the least side, $R$ the radius of the circum-circle.

Then $bc = a \cdot 2R$, $\therefore \frac{bc}{2a} = R = \frac{abc}{2S}$, $\therefore S = \frac{a^2}{2}$.

4. Let $A'$, $B'$, $C'$ be the feet of the perpendiculars from $D$ on $BC$, $CA$, $AB$.

Then $DA' \cdot BC = 2 \triangle BDC = DC \cdot DB \sin A$,

$$\therefore \frac{1}{DA'} = \frac{BC}{\sin A} \cdot \frac{1}{DB \cdot DC} = \frac{2R \cdot DA}{DA \cdot DB \cdot DC} = \frac{DA}{\mu} \text{ suppose}$$

$$\therefore DA \cdot DA' = \mu = DB \cdot DB' = DC \cdot DC' \text{ by symmetry.}$$

Now $A'C'^2 = DA'^2 + DC'^2 - 2DA' \cdot DC' \cdot \cos A'DC$

$$= \mu^2 \left\{\frac{1}{DA^2} + \frac{1}{DC^2} + \frac{2\cos B}{DA \cdot DC}\right\}$$

$$= \frac{\mu^2}{DA^2 \cdot DC^2}(DA^2 + DC^2 + 2\,DA \cdot DC\cos B) = \frac{\mu^2 AC^2}{DA^2 \cdot DC^2},$$

$$\therefore A'C' = \frac{\mu AC}{DA \cdot DC}.$$

Now $d \cdot A'C' = DA' \cdot DC' \sin B = \dfrac{\mu^2}{DA \cdot DC} \sin B,$

$$\therefore d = \frac{\mu^2}{DA \cdot DC} \sin B \times \frac{DA \cdot DC}{\mu \cdot AC} = \frac{\mu \cdot \sin B}{AC} = \frac{DA \cdot DB \cdot DC}{2R} \cdot \frac{1}{2R},$$

$$\therefore 4dR^2 = DA \cdot DB \cdot DC.$$

5. Produce $PC$ to meet the curve in $P'$, and join $PQ$, $P'Q'$. Then since $PC = CP'$, and $QC = CQ'$, $\therefore$ $PQP'Q'$ is a parallelogram.

$\therefore$ the angle $TPQ = PP'Q = P'PQ'$. To each add $CPT$. $\therefore$ the angle $CPQ = TPQ'$.

6. Draw $SY$, $SZ$, $S'Y'$, $S'Z'$ perpendiculars from the foci on the tangents. Then the angle $SPY = S'PY'$,

$$\therefore \sin^2 SPT = \sin SPY \cdot \sin S'PY' = \frac{SY}{SP} \cdot \frac{S'Y'}{S'P} = \frac{BC^2}{CD^2}$$

$$\sin^2 SQT = \sin SQT \cdot \sin S'QZ = \frac{SZ}{SQ} \cdot \frac{S'Z'}{S'Q} = \frac{BC^2}{CD'^2},$$

$$\therefore \sin^2 SPT + \sin^2 SQT = BC^2 \left( \frac{1}{CD^2} + \frac{1}{CD'^2} \right) = \text{const. since } CD \text{ and}$$

$CD'$ are at right angles. Salm. *Con.* Art. 159, Ex. 1.

7. If $f$ be the acceleration of $W$, the acceleration of $P$ will be $nf$.

The kinetic energy of the system at the end of time $t$ is

$$\tfrac{1}{2} P n^2 f^2 t^2 + \tfrac{1}{2} W f^2 t^2. \quad \ldots\ldots\ldots \quad (A)$$

The work done by gravity during the time $t$ is

$$\tfrac{1}{2} \left( P - \frac{W}{n} \right) nfgt^2. \quad \ldots\ldots\ldots \quad (B)$$

$\therefore$ equating $(A)$ and $(B)$ we have

$$f = \frac{\left( P - \frac{W}{n} \right) n}{n^2 P + W} \cdot g = \frac{nP - W}{n^2 P + W} \cdot g.$$

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## Paper XCVII.

1. Tripos 1875. Wednesday morning. No. 5.

2. $2\sin^3\theta = \sin(A-\theta)\{\cos(B-C) - \cos(B+C-2\theta)\}$,

$\therefore 4\sin^3\theta = \sin(A+B-C-\theta) + \sin(A+C-B-\theta)$
$\quad - \sin(B+C-A-\theta) - \sin(A+B+C-3\theta)$
$= \sin(2C+\theta) + \sin(2B+\theta) + \sin(2A+\theta) - \sin 3\theta$,

$\therefore 3\sin\theta = 4\sin^3\theta + \sin 3\theta$
$= \sin\theta(\cos 2C + \cos 2B + \cos 2A)$
$\quad + \cos\theta(\sin 2C + \sin 2B + \sin 2A)$,

$\therefore \cot\theta = \dfrac{3 - \cos 2A - \cos 2B - \cos 2C}{\sin 2A + \sin 2B + \sin 2C}$

$= \dfrac{4(1 + \cos A\cos B\cos C)}{4\sin A\sin B\sin C}$, Todh. *Trig.* Cap. viii. Ex. 18.

$= \cot A + \cot B + \cot C$. „ „ „ Ex. 28.

3. Consider $T$ as a point circle. Then $R$ is a point on the radical axis.

4. Tripos 1878. Monday morning. No. 10.

5. The equation to the normal to the parabola $y^2 = lx$ at the point $(x', y')$ is

$$y - y' = -\frac{2y'}{l}(x - x').$$

If $(h, k)$ be a point through which this normal passes,

$$k - y' = -\frac{2y'}{l}(h - x').$$

The equation which determines the ordinates of the points in which normals through $(h, k)$ meet the parabola is

$$k - y' = -\frac{2y'}{l}\left(h - \frac{y'^2}{l}\right),$$

or
$$y'^3 + \left(\frac{l}{2} - h\right)ly' - \frac{l^2k}{2} = 0,$$

$\therefore$ if $y_1, y_2, y_3$ be the roots of this equation,

$$y_1 + y_2 + y_3 = 0,\ y_2y_3 + y_3y_1 + y_1y_2 = l\left(\frac{l}{2} - h\right),\ y_1y_2y_3 = \frac{l^2k}{2}.$$

Now twice the area of the triangle $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$,

$$2\triangle = y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2)$$

$$= \frac{1}{l}\left\{y_1(y_2^2 - y_3^2) + y_2(y_3^2 - y_1^2) + y_3(y_1^2 - y_2^2)\right\}$$

$$= \frac{1}{l}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1).$$

Now $$y_3 = -(y_1 + y_2),$$

$$\therefore l\left(\frac{l}{2} - h\right) = y_1y_2 + y_3(y_1 + y_2) = y_1y_2 - (y_1 + y_2)^2 = -3y_1y_2 - (y_1 - y_2)^2,$$

$$\therefore (y_1 - y_2)^2 = -3y_1y_2 - l\left(\frac{l}{2} - h\right).$$

Similarly $$(y_2 - y_3)^2 = -3y_2y_3 - l\left(\frac{l}{2} - h\right),$$

and $$(y_3 - y_1)^2 = -3y_3y_1 - l\left(\frac{l}{2} - h\right),$$

$$\therefore -4l^2\triangle^2 = -(y_1 - y_2)^2(y_2 - y_3)^2(y_3 - y_1)^2$$

$$= \left\{3y_1y_2 + l\left(\frac{l}{2} - h\right)\right\}\left\{3y_2y_3 + l\left(\frac{l}{2} - h\right)\right\}\left\{3y_3y_1 + l\left(\frac{l}{2} - h\right)\right\}$$

$$= 27\,y_1^2y_2^2y_3^2 + 9l\left(\frac{l}{2} - h\right)y_1y_2y_3(y_1 + y_2 + y_3)$$

$$+ 3l^2\left(\frac{l}{2} - h\right)^2(y_1y_2 + y_2y_3 + y_3y_1) + l^3\left(\frac{l}{2} - h\right)^3$$

$$= \frac{27l^4k^2}{4} + 3l^3\left(\frac{l}{2} - h\right)^3 + l^3\left(\frac{l}{2} - h\right)^3,$$

$$\therefore -\frac{16\triangle^2}{27l^2} = k^2 + \frac{2}{27l}(l - 2h)^3.$$

Now since $(h, k)$ lies on the curve

$$y^2 + \frac{2}{27l}(l - 2x)^3 = -a^2, \text{ we see that}$$

$$\frac{16\triangle^2}{27l^2} = a^2, \therefore \triangle = \frac{3al\sqrt{3}}{4} = \text{const.}$$

If the point $(h, k)$ be taken so that two of the normals coincide, the area of the triangle will vanish, and $\therefore a = 0$.

$\therefore$ the locus of the point from which only two separate normals can be drawn to the parabola $y^2 = lx$, is the curve

$$y^2 + \frac{2}{27l}(l - 2x)^3 = 0.$$

6. If $(h, k)$ be the point of intersection of the normals, the points where they meet curve $y^2 = lx$ are given by the intersection of the parabola with the curve

$$k - y = -\frac{2y}{l}(h - x),$$

or

$$2xy = y(2h - l) + lk.$$

This represents a rectangular hyperbola whose asymptotes are parallel to the axes, and whose centre is given by the equations $2x = h - l$, $2y = 0$. The latter shews that the centre always lies on the axis of the parabola.

7. Tripos 1875. Tuesday morning. No. 12.

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## Paper XCVIII.

1. Let $I$ be $A$'s income for any given year.

First suppose $I$ to increase in A.P. common difference $b$.

Then $I + b$, $I + 2b$, $I + 3b$, $I + 4b$, $I + 5b$ . . . represents his income for the 2nd, 3rd, 4th, 5th, 6th . . . years. $\therefore$ if $c$ denote the percentage, the income tax on the 3rd, 4th, 5th, 6th . . . years is

$$\frac{I + b}{c}, \frac{I + 2b}{c}, \frac{I + 3b}{c}, \frac{I + 4b}{c} \ldots,$$

which is an A.P. the common difference being $\frac{b}{c}$.

Next let the income increase in G.P., the common ratio being $r$. Then his income for the different years beginning with the first is $I$, $Ir$, $Ir^2$ . . . $\therefore$ if $d$ be the percentage, the tax for the 3rd, 4th, 5th ... years is

$$\frac{I + Ir + Ir^2}{3d}, \frac{Ir + Ir^2 + Ir^3}{3d}, \frac{Ir^2 + Ir^3 + Ir^4}{3d} \quad \ldots \ldots$$

which is a G.P. common ratio $r$.

2. $u_2 = 2u_1k,$

$u_3 = 3\{(1+k)u_2 - 2u_1k\} = 3u_2k = 3\,.\,2\,.\,u_1k^2,$

$u_4 = 4\{(1+k)u_3 - 3u_2k\} = 4u_3k = 4\,.\,3\,.\,2\,.\,u_1k^3,$ &c.

$$\therefore \frac{u_1}{0!1!} + \frac{u_2}{1!2!} + \frac{u_3}{2!3!} + \ldots$$

$$= u_1 + u_1k + u_1\frac{k^2}{2!} + u_1\frac{k^3}{3!} + \ldots$$

$$= u_1\left(1 + k + \frac{k^2}{2!} + \frac{k^3}{3!} + \ldots\right) = u_1e^k\,.$$

3. Let $S = \sin\theta\,.\,\frac{\sin\theta}{1} - \sin 2\theta\,.\,\frac{\sin^2\theta}{2} + \sin 3\theta\,.\,\frac{\sin^3\theta}{3}\ldots$

$$C = \cos\theta\,.\,\frac{\sin\theta}{1} - \cos 2\theta\,.\,\frac{\sin^2\theta}{2} + \cos 3\theta\,.\,\frac{\sin^3\theta}{3}\ldots$$

$$\therefore C + Si = e^{\theta i}\,.\,\frac{\sin\theta}{1} - e^{2\theta i}\,.\,\frac{\sin^2\theta}{2} + e^{3\theta i}\,.\,\frac{\sin^3\theta}{3}\ldots$$

$$= \log(1 + e^{\theta i}\sin\theta),$$

$$\therefore 1 + e^{\theta i}\sin\theta = e^{C+Si} = e^C\,.\,e^{Si},$$

$$\therefore 1 + \sin\theta(\cos\theta + i\sin\theta) = e^C\,.\,(\cos S + i\sin S),$$

$$\therefore e^C\cos S = 1 + \sin\theta\cos\theta \quad (1)$$

$$e^C\sin S = \sin^2\theta \quad (2)$$

$$\therefore \cot S = \operatorname{cosec}^2\theta + \cot\theta = 1 + \cot\theta + \cot^2\theta.$$

Also squaring and adding (1) and (2)

$$e^{2C} = 1 + \sin 2\theta + \sin^2\theta$$

$$\therefore 2C = \log\{1 + \sin 2\theta + \sin^2\theta\}.$$

4. Let $I$ be the centre of the inscribed circle. Draw $ID$, $OE$ perpendicular to $AB$. Then $D$ and $E$ are points of contact and $AE = s$, $AD = s - a$, where $2s = a + b + c$.

Then $AE : AD :: OA : IA,$

$$\therefore s : s - a :: OA^2 : OA\,.\,IA$$

$$:: OA^2 : bc,$$ from similar triangles $OAB$, $IAC$.

And from the same triangles,

$$OB : IC :: OA : AC,$$

$$\therefore \frac{OB^2}{bc} = \frac{IC^2}{ab} \cdot \frac{OA^2}{bc} = \frac{s-c}{s} \cdot \frac{s}{s-a} = \frac{s-c}{s-a}.$$

Similarly $$\frac{OC^2}{ab} = \frac{s-b}{s-a},$$

$$\therefore \frac{OA^2}{bc} - \frac{OB^2}{ca} - \frac{OC^2}{ab} = 1.$$

5. Let $A$, $A'$, $O$ be the vertices of the ellipse and cone, and let the focal spheres touch $OA$ in $E$ and $E'$.

Then $$A'O - AO = OF' - A'F' - (OE + EA)$$
$$= OF' - A'F' - OF - AS$$
$$= A'F - AS = A'S - AS = SS'.$$

$\therefore$ the locus of $O$ is a hyperbola, whose foci are $A$ and $A'$.

6. Tripos 1878. Tuesday morning. No. 6.

7. On $AB$ as diameter describe a circle, centre $O$. Draw $PN$ perpendicular to $AB$ meeting the circle in $R$. Let $AOR = \theta$. Then the tangent at $P$ makes an angle $\frac{\theta}{2}$ with the horizon.

The length $PQ$ = arc $AP$ = 2 . chord $AR$, and $PQ$ is parallel to $AR$.

If $t$ and $t'$ be the times down $PQ$ and $AR$,

$$t^2 = \frac{2PQ}{g \sin \frac{\theta}{2}} = \frac{4AR}{g \sin \frac{\theta}{2}}; \quad t'^2 = \frac{2AR}{g \sin \frac{\theta}{2}}$$

$$\therefore t = \sqrt{2} \, . \, t'.$$

Now $t'$ is constant, being the time of descent to the lowest point of a vertical circle. $\therefore$ $t$ is constant.

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## Paper XCIX.

1. (1). Let $$X = (x^3 + 3x^2 + 34x + 37)^{\frac{1}{3}};$$
$$Y = (x^3 - 3x^2 + 34x - 37)^{\frac{1}{3}}.$$

Cube both sides of the given equation.

$$\therefore 8 = X^3 - Y^3 - 3XY(X - Y), \text{ and } X - Y = 2,$$

$$\therefore x^2 + 11 = XY = \{(x^3 + 34x)^2 - (3x^2 + 37)^2\}^{\frac{1}{3}},$$

$$\therefore x^6 + 33x^4 + 363x^2 + 1331 = x^6 + 59x^4 + 934x^2 - 1369,$$

$$\therefore 26x^4 + 571x^2 - 2700 = 0,$$

$$\therefore x^2 = 4 \text{ or } -\tfrac{625}{26}; \ \therefore x = \pm 2, \pm 15\sqrt{-\tfrac{2}{26}}.$$

(2). Multiply (1) by $y$, and (2) by $x$, and then subtract.

$$x^4y - 8y^4x + 48y(y^2 - 1) = 0$$

$$x^4y - 8y^4x - 6x(x^2 - 4) = 0,$$

$$\therefore 8y^3 - x^3 - 8y + 4x = 0, \quad \therefore (2y - x)(4y^2 + 2xy + x^2 - 4) = 0.$$

Put $x = 2y$ in the given equations. $\therefore y = \pm 1, x = \pm 2.$

Again, add the given equations.

$$\therefore x^3(x + 2y) - 8y^3(x + 2y) - 12(x^2 - 4y^2) = 0,$$

$$\therefore (x + 2y)(x - 2y)(x^2 + 2xy + 4y^2 - 12) = 0.$$

Put $x = -2y$. $\therefore y = \dfrac{-3 \pm \sqrt{33}}{4}, \ x = \dfrac{-3 \pm \sqrt{33}}{2}.$

(3). Put $1 - 16y^2 = u, \ 1 - 16x^2 = v.$

$$\left.\begin{aligned} \therefore \sqrt{u} - \sqrt{v} &= 2(x + y) \\ \therefore \sqrt{u} + \sqrt{v} &= 8(x - y) \end{aligned}\right\} \text{ and } u - v = 16(x^2 - y^2).$$

Square and add.

$$\therefore 2(u + v) = 4(x + y)^2 + 64(x - y)^2,$$

or $$1 - 8x^2 - 8y^2 = (x + y)^2 + 16(x - y)^2.$$

Let $x + y = \alpha, \ x - y = \beta. \ \therefore 1 = 5\alpha^2 + 20\beta^2$

Also from the 2nd given equation, $\frac{2}{5} = 3\alpha^2 - \beta^2$

$$\therefore \alpha^2 = \tfrac{9}{65}, \beta^2 = \tfrac{1}{65}. \ \therefore x + y = \pm \frac{3}{\sqrt{65}}, \ x - y = \pm \frac{1}{\sqrt{65}}.$$

From these equations we can obtain the four values of $x$ and $y$.

2. Let the inscribed circle touch $AC$ in $E$. Then $OE$ is perpendicular to $AC$.

$$\therefore AE = AO \cos \frac{A}{2}, \ EC = OE \cot \frac{C}{2} = OA \sin \frac{A}{2} \cot \frac{C}{2},$$

$$\therefore\ b = AE + EC = AO\left(\cos\frac{A}{2} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\sin\frac{A}{2}\right) = AO\frac{\cos\frac{B}{2}}{\sin\frac{C}{2}},$$

$$\therefore\ AO = b\sec\frac{B}{2}\sin\frac{C}{2}.\quad \text{See Errata.}$$

$$\text{Now } ax = \frac{ar}{\sin\frac{A}{2}} = 4Rr\cos\frac{A}{2}, \text{ \&c.} \quad \ldots\ldots\ldots\ldots\ (1)$$

$$\therefore\ (A) = 2(b^2c^2y^2z^2 + c^2a^2z^2x^2 + a^2b^2x^2y^2) - a^4x^4 - b^4y^4 - c^4z^4$$

$$= (ax + by + cz)(by + cz - ax)(cz + ax - by)(ax + by - cz)$$

$$= (4Rr)^4\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)\left(\cos\frac{B}{2} + \cos\frac{C}{2} - \cos\frac{A}{2}\right)$$

multiplied by 2 other factors.

$$\text{Now } \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{\pi - A}{4}\cos\frac{\pi - B}{4}\cos\frac{\pi - C}{4}$$

Todh. *Trig.* Cap. viii. Ex. 20.

$$= 4\cdot\frac{1}{\sqrt{2}}\left(\cos\frac{A}{4} + \sin\frac{A}{4}\right)\frac{1}{\sqrt{2}}\left(\cos\frac{B}{4} + \sin\frac{B}{4}\right)\frac{1}{\sqrt{2}}\left(\cos\frac{C}{4} + \sin\frac{C}{4}\right),$$

$$\text{and } \cos\frac{B}{2} + \cos\frac{C}{2} - \cos\frac{A}{2} = 4\cos\frac{\pi - A}{4}\cos\frac{\pi + B}{4}\cos\frac{\pi + C}{4}.$$ *Id.* Ex. 21.

$$= 4\cdot\frac{1}{\sqrt{2}}\left(\cos\frac{A}{4} + \sin\frac{A}{4}\right)\frac{1}{\sqrt{2}}\left(\cos\frac{B}{4} - \sin\frac{B}{4}\right)\frac{1}{\sqrt{2}}\left(\cos\frac{C}{4} - \sin\frac{C}{4}\right).$$

$$\therefore\ (A) = 4(4Rr)^4\left(\cos^2\frac{A}{4} - \sin^2\frac{A}{4}\right)^2\left(\cos^2\frac{B}{4} - \sin^2\frac{B}{4}\right)^2\left(\cos^2\frac{C}{4} - \sin^2\frac{C}{4}\right)^2$$

$$= 4(4R)^4r^4\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}$$

$$= 4(4R)^4r^4\cdot\frac{ax}{4Rr}\cdot\frac{by}{4Rr}\cdot\frac{cz}{4Rr}, \text{ from (1)}$$

$$= (a + b + c)^2x^2y^2z^2, \text{ since } 4R = \frac{abc}{S}, \text{ and } r = \frac{S}{s}.$$

3. Let $ABCD$ be the quadrilateral. Let $E$, $F$, $G$ be the middle points of $BD$, $AC$, $EF$, and let $H$, $L$, $K$, $M$ be the middle points of $BC$, $CD$, $DA$, $AB$. Then since $AK = KD$, and $AM = MB$, $\therefore$ $KM$ is parallel to $BD$.

$\therefore$ $HLKM$ is a parallelogram having its sides parallel to $AC$ and $BD$.

Similarly $EHFK$ is a parallelogram having its sides parallel to $AB$ and $CD$.

Since the diagonals of a parallelogram bisect each other, $\therefore$ the middle points of $HK$ and $LM$ are at $G$.

Then by Todh. *Euc. Ap.* Prop. 1, $GA^2 + GB^2 = 2(AM^2 + GM^2)$, &c.

$$\therefore\ 2(GA^2 + GB^2 + GC^2 + GD^2) = HK^2 + LM^2 + \tfrac{1}{2}(AB^2 + BC^2 + CD^2 + DA^2).$$

Now $AB^2 + BC^2 = 2(AF^2 + BF^2)$, $CD^2 + DA^2 = 2(AF^2 + FD^2)$,

$$\therefore\ AB^2 + BC^2 + CD^2 + DA^2 = 4AF^2 + 2(BF^2 + FD^2)$$

$$= AC^2 + 4(BE^2 + EF^2) = AC^2 + BD^2 + 4EF^2,$$

$$\therefore\ 2(GA^2 + GB^2 + GC^2 + GD^2) = HK^2 + LM^2 + 2EF^2 + \tfrac{1}{2}(AC^2 + BD^2). \quad (1)$$

Again, $GA^2 + GC^2 = 2AF^2 + 2GF^2$; $GB^2 + GD^2 = 2BE^2 + 2GE^2$,

$$\therefore\ GA^2 + GB^2 + GC^2 + GD^2 = \tfrac{1}{2}(AC^2 + BD^2) + EF^2. \quad (2)$$

$\therefore$ subtracting (2) from (1) we have

$$GA^2 + GB^2 + GC^2 + GD^2 = HK^2 + LM^2 + EF^2.$$

4. Tripos 1878. Wednesday morning. No. 9.

5. Let $S$, $S'$ be the foci of the fixed ellipse, $C$ its centre, $H$ the other focus of the moving ellipse, and let $PQ$ be the chord of intersection. Join $P$ and $Q$ to $S$, $S'$, and $H$. Join $S'H$ cutting $PQ$ in $Z$.

Then $SP + S'P = 2a = SP + HP$, $\therefore$ $S'P = HP$. Similarly $S'Q = HQ$.

$\therefore$ $S'PH$, $S'QH$ are isosceles triangles. $\therefore$ $PQ$ bisects $S'H$ at right angles in $Z$.

Now $H$ lies on a fixed circle, centre $S$, and $S'$ is a fixed point, $\therefore$ by Todh. *Euc. Ap.* p. 332, the locus of $Z$ is a circle, centre $C$; and since $PQ$ is at right angles to $S'Z$ at the point $Z$, $\therefore$ the envelope of $PQ$ is a conic of which the locus of $Z$ is the auxiliary circle. Also, $C$ is the centre and $S'$ one of the foci. $\therefore$ the other focus is $S$.

It will be seen from a figure that if $SH < SS'$ $PQ$ will cut $SS'$ between $S$ and $S'$, and the locus is a hyperbola. If $SH > SS'$, $PQ$ will cut $SS'$ produced, and the locus is an ellipse. If $SH = SS'$, $PQ$ will always pass through the focus $S$, which may be considered as a point ellipse. See Errata.

6. On $AC$ take a point $E$ such that $EC = OA$, and on $BD$ take $F$ so that $FD = BO$.

The C. of G. of $\frac{OA}{AC}$ at $A$ and $\frac{OC}{AC}$ at $C$ is $E$, since $AE \cdot \frac{OA}{AC} = EC \cdot \frac{OC}{AC}$, and the weight which we may suppose placed at $E$ will be

$$\frac{OA + OC}{AC} = 1.$$

Similarly the C. of G. of $\frac{OD}{DB}$ at $D$, and $\frac{OB}{BD}$ at $B$ is $F$, and the weight $= 1$.

$\therefore$ instead of 5 particles at $A$, $B$, $C$, $D$, $O$, we have now 3 equal particles at $O$, $E$, $F$, and their C. of G. will be the C. of G. of the triangle $OEF$.

We will now find the C. of G. of the quadrilateral.

Bisect $OE$ in $G$, so that $G$ is the middle point of $AC$. Join $BG$, $GD$, and take points $K$, $L$ on $BG$ and $DG$ so that $KG = \frac{1}{3}BG$, $GL = \frac{1}{3}DG$, and join $KL$, which is parallel to $BD$.

Then $K$ is the C. of G. of the triangle $ABC$, and $L$ is the C. of G. of $ADC$.

$\therefore$ $M$ is the C. of G. of the quadrilateral $ABCD$, so that

$$\frac{LM}{MK} = \frac{\Delta ABC}{\Delta ADC} = \frac{BO}{OD} = \frac{DF}{FB}.$$

$\therefore$ $GM$ produced passes through $F$, and the C. of G. of $ABCD$ lies on $GF$. Similarly it can be shewn to lie on $EH$. $\therefore$ it is the point of intersection of $GF$ and $HE$, and $\therefore$ coincides with the C. of G. of the triangle $OEF$, *i.e.* it coincides with the C. of G. of the 5 particles at $A$, $B$, $C$, $D$, $O$, by what has been shewn.

7. $v_{r+1} = v_r + v_{r-1}$, $\therefore$ $v_r = v_{r+1} - v_{r-1}$;

$v_r = v_{r-1} + v_{r-2}$, $\therefore$ $v_r - v_{r-2} = v_{r-1}$;

$\therefore$ $v_r(v_r - v_{r-2}) = v_{r-1}(v_{r+1} - v_{r-1})$;

$\therefore$ $v_r^2 - v_r v_{r-2} = v_{r-1} v_{r+1} - v_{r-1}^2$;

$\therefore$ $v_r^2 \sim v_{r-1} v_{r+1} = v_{r-1}^2 \sim v_{r-2} v_r$

$= v_{r-2}^2 \sim v_{r-3} v_{r-1}$

. . . .

$= v_2^2 \sim v_1 v_3$, and $v_3 = v_2 + v_1$,

$= \lambda^2 v_1^2 - (\lambda + 1) v_1^2$, for $v_2 = \lambda v_1$,

$= v_1^2 (\lambda^2 - \lambda - 1)$.

PAPER C.

1. From (1) and (2) we have

$$zx - yz = \frac{x^3}{y+z} - \frac{y^3}{z+x},$$

$$\therefore z(x-y)(y+z)(z+x) = x^3(z+x) - y^3(y+z)$$

$$= z(x^3 - y^3) + x^4 - y^4,$$

$\therefore z(y+z)(z+x) = z(x^2+xy+y^2) + (x+y)(x^2+y^2)$, since $x \neq y$,

$\therefore$ reducing, we have

$$xz^2 - x^2z + yz^2 - y^2z = x^3 + x^2y + xy^2 + y^3 - z^3. \quad . \quad (A)$$

Similarly from (2) and (3) we obtain on reduction

$$x^2z - xz^2 + x^2y - xy^2 = y^3 + y^2z + yz^2 + z^3 - x^3. \quad . \quad (B)$$

Adding $(A)$ and $(B)$ we have

$$xy^2 + y^3 + y^2z = 0, \text{ and } y \neq 0,$$

$$\therefore x + y + z = 0.$$

Again $yz + \frac{x^3}{y+z} = yz - x^2$, since $y + z = -x$

$$= yz + x(y+z) \quad ,, \quad ,,$$

$$= yz + xy + xz.$$

2. Tripos 1878. 2nd Monday afternoon. No. 2.

3. Let 1, 2, 3, 4, 5, 6 be the feet of the perpendiculars from $O$ on $DC$, $DB$, $DA$, $CA$, $CB$, $BA$. Then 134 is the pedal line of the triangle $ADC$, 125 of $BCD$, 236 of $ABD$, 456 of $ABC$.

The circle on $OD$ as diameter evidently passes through 1, 2, 3, *i.e.* the circle through 1, 2, 3 passes through $O$. Similarly we can see that $O$ lies on each circle described round the triangles formed by taking any three of the six points 1, 2, 3, 4, 5, 6. That is the point $O$ is the intersection of the circles round the four triangles formed by the four pedal lines. $\therefore$ if we describe a parabola touching these four pedal lines, the focus will be at $O$, and the feet of the perpendiculars from $O$ on the four pedal lines will lie on a straight line, viz. the tangent at the vertex of the parabola. See LXXXVII. No. 5.

This question is solved analytically in Smith's *Conics*, p. 83. From the form of the solution there given we see that if $O$ be fixed, and any number of points be taken on the circumference, and triangles formed by joining these points by threes, and the pedal lines of all these triangles be drawn with regard to $O$, the feet of the perpendiculars from $O$ on these pedal lines are collinear.

4. Let $S$ and $H$ be the foci of the fixed ellipse, $S'$ the other focus of the moving ellipse. Let $TP'P$, $TQ'Q$ be two common tangents, $P$ and $Q$ being on the fixed ellipse.

Then the angle $HTQ = STP = S'TQ$. $\therefore$ $H$, $S'$, $T$ are collinear.

Draw $SY$, $HZ$, $S'Z'$ perpendicular to $TP$, and let $b$, $b'$ be the semi-minor axes.

Then $HZ \cdot SY = b^2$, $S'Z' \cdot SY = b'^2$, $\therefore$ $HT : S'T$ in a constant ratio. $\therefore$ $HS' : HT$ in a constant ratio. Now the locus of $S'$ is a circle, $\therefore$ the locus of $T$ is a circle. Todh. *Euc. Ap.* p. 332.

5. Let $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ be the coordinates of $P$, $Q$, $R$, and let the equation to the parabola be $y^2 = 4ax$.

Draw the ordinates $PN$, $pn$. Then $PN \cdot pn = 4a^2$.

$\therefore$ the equation to the diameter $Ap$ is $y = -\dfrac{4a^2}{y_1}$

The equation to $QR$ is

$$y - y_3 = \frac{y_2 - y_3}{x_2 - x_3}(x - x_3) = \frac{4a(y_2 - y_3)}{y_2^2 - y_3^2}(x - x_3) = \frac{4a}{y_2 + y_3}(x - x_3),$$

$\therefore$ at $A$, $-\dfrac{4ax}{y_2 + y_3} = -\dfrac{y_3^2}{y_2 + y_3} + \dfrac{4a^2}{y_1} + y_3 = \dfrac{y_2 y_3}{y_2 + y_3} + \dfrac{4a^2}{y_1}$.

$\therefore$ the equation to $AS$ is

$$y\left\{\frac{y_2 y_3}{4a} + \frac{a(y_2 + y_3)}{y_1} + a\right\} = \frac{4a^2}{y_1}(x - a)$$

or
$$16a^3(x - a) = y\{y_1 y_2 y_3 + a^2(y_1 + y_2 + y_3)\}.$$

The symmetry of this equation shews that we should obtain the same equation for the lines $BS$, and $CS$. $\therefore$ $A$, $B$, and $C$ lie on a straight line passing through the focus.

6. Let $A$ denote the area of the first lamina, $a$ the distance of its centre of gravity from the vertical line through the centres.

Then since the thickness of each is the same, the masses are proportional to the areas, and $\therefore$ to the squares of the radii.

$\therefore$ the distance of the centre of gravity of the pile from the vertical line

$$= \frac{Aa\{1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3\}}{A\{1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2\}}.$$

Now $(2n-1)^3 = (2n-1)\{(2n-1)^2 - 4\} + 4(2n-1)$

$$= (2n-3)(2n-1)(2n+1) + 4(2n-1),$$

$\therefore$ the sum of $n$ terms of the series in the numerator

$$= \frac{(2n-3)(2n-1)(2n+1)(2n+3)}{4 \cdot 2} \quad \frac{3 \cdot 3}{4 \cdot 2} + 4n^2$$

$$= \frac{(4n^2-9)(4n^2-1) - 9}{8} + 4n^2$$

$$= n^2(2n^2 - 1).$$

And $(2n-1)^2 = 4n(n-1) + 1.$

$\therefore$ the sum of $n$ terms of the series in the denominator

$$= \frac{4(n-1)n(n+1)}{3} + n$$

$$= \frac{n}{3}(4n^2 - 1)$$

$\therefore$ the required ratio is $3n(2n^2-1) : 4n^2 - 1.$

7. Let $CA$, $CB$ be sections of the given plane made by a vertical plane passing through the pulley, and at right angles to the common section, and let $A$ and $B$ be the positions of the particles at any time.

Then if $G$ be the C. of G. of $A$ and $B$, $m$ and $n$ the masses of $A$ and $B$,

$$AG : GB :: n : m.$$

Take $CA$ and $CB$ as axes of $x$ and $y$, and draw $GM$, $GN$ parallel to them, meeting $CA$ in $N$ and $CB$ in $M$. Then from similar triangles

$$CA : CN :: AB : BG :: m + n : m,$$

$$\therefore CA = (m+n)\frac{x}{m}. \quad \text{Similarly } CB = (m+n)\frac{y}{n}.$$

$$\therefore l = CA + CB = (m+n)\frac{x}{m} + (m+n)\frac{y}{n}$$

$$\therefore \frac{x}{m} + \frac{y}{n} = \frac{l}{m+n},$$

which is the equation to a straight line, and is the locus of $G$.

Let $T$ be the tension of the string, and suppose $A$ to be moving up the plane.

Then the acceleration of $A$ upwards is $\frac{T}{m} - g \sin \alpha$,

and " $B$ downwards is $g \sin \beta - \frac{T}{n}$,

and these are equal, $\therefore \frac{T}{m} - g \sin \alpha = g \sin \beta - \frac{T}{n}$.

$$\therefore T = g \cdot \frac{mn}{m+n} (\sin \alpha + \sin \beta).$$

$\therefore$ if $f$ be the acceleration of $A$,

$$f = \frac{g}{m+n} \{n \sin \alpha + n \sin \beta - (m+n) \sin \alpha\}$$

$$= \frac{g}{m+n} (n \sin \beta - m \sin \alpha).$$

If $EF$ be the locus of $G$, we may for simplicity suppose $w_1$ to be at $E$ and $w_2$ at $F$.

Then the acceleration of $w_1$ along $EF$

$$= \frac{f}{w_1} \sqrt{w_1^2 + w_2^2 - 2w_1 w_2 \cos ECF}$$

$$= \frac{f}{w_1} \sqrt{w_1^2 + w_2^2 + 2w_1 w_2 \cos (\alpha + \beta)} = F_1 \text{ suppose,}$$

with a similar expression for $F_2$.

And, by Todh. *Mech.* p. 295, the acceleration of the C. of. G.

$$= \frac{w_1 F_1 + w_2 F_2}{w_1 + w_2}$$

$$= 2g \frac{w_1 \sin \alpha \sim w_2 \sin \beta}{(w_1 + w_2)^2} \sqrt{w_1^2 + w_2^2 + 2w_1 w_2 \cos (\alpha + \beta)}.$$

# APPENDIX.

# APPENDIX.

## ADDITIONAL PROBLEMS.

### PAPER VI.

7. An ellipse is described having for axes the tangent and normal at any point $P$ of a fixed ellipse, and touching one of the axes of the fixed ellipse at its centre. Prove that the locus of the focus of the moving ellipse is two circles, of radii $a \pm b$.

### PAPER VII.

5. If $AA'BB'$, $BB'CC'$, $CC'AA'$ be three circles, and the straight lines $AA'$, $BB'$, $CC'$ cut the circle $A'B'C'$ again in $\alpha$, $\beta$, $\gamma$, respectively, the triangle $\alpha\beta\gamma$ will be similar to $ABC$.

### PAPER XI.

6. Prove that the asymptotes of the curve

$$11x^2 + 24xy + 4y^2 - 2x + 16y + 11 = 0,$$

are given by the equation

$$(11x + 2y + 9)(x + 2y - 1) = 0.$$

Trace the curve, find the lengths of its axes, and prove that the equation of its director circle is

$$x^2 + y^2 + 2x - 2y = 1.$$

### PAPER XII.

7. $AC$, $CB$ are chords at right angles in a circle, $P$ is any point on the circumference. $PA$, $PB$, $PC$ represent forces. Shew that the locus of the extremity of the straight line which represents their resultant is a circle.

## Paper XVII.

7. Along the sides of a regular hexagon taken in order act 6 forces represented by 1, 2, 3, 4, 5, 6 respectively. Prove that their resultant will be represented by 6, and that its direction will be parallel to one of the sides, and at a distance from the centre of the hexagon equal to $3\frac{1}{2}$ times the radius of the inscribed circle.

## Paper XVIII.

1. A road runs from $A$ to meet another at right angles. Shew that there are two points on the second road which may be reached in the same time from $A$ whether we travel by road or across country, the rates of travelling by road and across country being as 7 : 5. Also shew that for places between these the quickest route is across country, and the quickest for all other places is by road.

2. If 16 be added to the product of four consecutive odd or even numbers, the result is always a square number. For odd numbers its last digit in four cases out of five is 1, in the remaining case 5. For even numbers the last digit in four cases out of five is 6, in the remaining case 0.

3. If $l$, $m$, $n$ be the distances of any point in the plane of a triangle $ABC$ from its angular points, and $d$ its distance from the circum-centre, shew that

$$l^2 \sin 2A + m^2 \sin 2B + n^2 \sin 2C = 4(R^2 + d^2) \sin A \sin B \sin C,$$

$R$ being the radius of the circum-circle.

7. A conic passes through the centres of the four circles which touch the sides of a triangle. Prove that the locus of its centre is the circumscribing circle.

## Paper XIX.

2. Solve the equations

(1) $x^4 + a^4 = 4ax(x^2 + a^2)$.

(2) $\frac{x}{a} + \frac{b}{y} + \frac{c}{z} = \frac{a}{x} + \frac{y}{b} + \frac{c}{z} = \frac{a}{x} + \frac{b}{y} + \frac{z}{c} = 1.$

4. If $R$ be the radius of the circum-circle, shew that the area of the triangle

$$= \tfrac{3}{2} R^2 \{\sin^3 A \cos (B - C) + \sin^3 B \cos (C - A) + \sin^3 C \cos (A - B)\}.$$

5. If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by two lines equally inclined to the bisector, one terminated by the base and the other by the circum-circle.

## PAPER XX.

3. If the area of a quadrilateral be $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, shew that it can be inscribed in a circle.

4. $AB$, $CD$ are chords of a circle intersecting in $O$, and $AC$, $DB$ meet at $P$. If circles be described about the triangles $AOC$, $BOD$, the angle between their tangents at $O$ will be equal to $APB$, and their other common point will lie on $OP$.

5. A uniform rod $AB$ rests with its ends on a rough circular wire in a vertical plane, and the equilibrium is limiting. Shew that the vertical through the centre of the rod meets the circle through $A$, $B$ and the centre of the wire in two points, in one of which the directions of the resultant actions at $A$ and $B$ meet.

6. $2a$ and $2b$ are the major and minor axes of an ellipse. With centre $O$ as centre, and radii $a$, $b$, $a+b$ circles are described, and a radius vector $OPQR$ is drawn meeting them respectively in $P$, $Q$, $R$. If a parallel to the minor axis drawn through $P$ meet a parallel to the major axis drawn through $Q$ in $S$, then $S$ is a point on the ellipse, and $SR$ is the normal at $S$.

7. Defining the angle at which two circles cut to be that in which no part of either circle lies, prove that if the circles

$$(x-b)(x-b') + y^2 = 0, \quad (x-a)(x-a') + y^2 = 0,$$

cut at an angle $\theta$,

$$(a-a')^2 (b-b')^2 \sin^2\theta + 4(b'-a)(b-a)(b'-a')(b-a') = 0.$$

## PAPER XXI.

1. The population of a town at the end of any year can be found by subtracting eleven times the population at the end of the previous year from ten times the population at the end of the succeeding year. Nine years ago the population was 1210, eleven years ago it was 1000. Prove that it increases in G.P.

2. Prove that $5\sin^{-1}\frac{1}{\sqrt{50}} + 2\sin^{-1}\frac{3}{25\sqrt{10}} = \frac{\pi}{4}$.

3. Sum to infinity the series

$$(1)\ \sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{2^2}\sin 3\theta + \frac{1}{2^3}\sin 4\theta + \ldots$$

$$(2)\ \sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \frac{1}{7}\sin 7\theta + \ldots$$

$$(3)\ \sin\theta + \frac{1}{4}\sin 3\theta + \frac{1}{4^2}\sin 5\theta + \frac{1}{4^3}\sin 7\theta + \ldots$$

4. The three perpendiculars from the angles $A$, $B$, $C$ of a triangle on the opposite sides meet the sides in $D$, $E$, $F$. If $D$, $E$, $F$ be given, shew how to construct the triangle $ABC$.

5. $P$ is the orthocentre of a triangle, $Q$ any point on the circumcircle. Shew that $PQ$ is bisected by the pedal line of the triangle with respect to the point $Q$.

6. If $\lambda$ be a variable parameter, the locus of the vertices of the hyperbolas represented by

$$x^2 - y^2 + \lambda xy = a^2$$

is the curve $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

## Paper XXII.

2. If the impossible root of $x^3 + qx + r = 0$ be $\alpha + \beta\sqrt{-1}$, shew that $\beta^2 = 3\alpha^2 + q$.

4. Through the angular points of a triangle $ABC$ draw straight lines perpendicular to the lines bisecting the angles. If $\Delta$, $P$ be the area and perimeter of the original triangle, $\Delta'$, $P'$ those of the new triangle, prove that

$$(1)\ 4\Delta\Delta' = Pabc; \quad (2)\ PP' = 4\Delta'\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right).$$

6. If $ABCD$ be a quadrilateral inscribed in a circle, and the sides be produced to meet in $F$ and $G$, prove that the bisectors of the angles at $F$ and $G$ meet at right angles.

7. Chords of a hyperbola are drawn through a fixed point. Shew that the locus of their middle points is a hyperbola similar to the original hyperbola or its conjugate.

## PAPER XXIII.

1. If $x\left(1 - \frac{mzy}{x^3}\right) = y\left(1 - \frac{mxz}{y^3}\right) = z\left(1 - \frac{myx}{z^3}\right)$,

and $x, y, z$ be unequal, prove that each member of these equations

$$= x + y + z - m.$$

2. A besieged garrison is provisioned for a certain number of days; after 10 days one-sixteenth of the men are killed in a sortie, when it is calculated that by diminishing the daily rations by one-fifth it will be able to hold out for 30 days longer than was first supposed. Subsequently 150 men with a quantity of provisions equal to half what is still left come in; by which it will be enabled to increase the time it can still hold out by one-fourth. How many men were there originally? and for how long was it provisioned?

## PAPER XXV.

6. Out of a wooden cylinder is cut a cone of the same base, and the hole is filled up with lead. If lead be nine times as heavy as wood, and if the centre of gravity of the whole be at the vertex of the cone, shew that

the height of the cone : the height of the cylinder :: sin 18° : 1.

## PAPER XXX.

1. Prove that (1) the coefficient of $x^{m+1}y^{n+1}$ in the expansion of

$$\frac{(1-x)(1-y)}{1-x-y} \text{ is } \frac{(m+n)!}{m!\,n!}.$$

(2) the coefficient of $x^{n-1}$ in the expansion of

$$\{(1-x)(1-cx)(1-c^2x)(1-c^3x)\}^{-1},$$

in ascending powers of $x$ is

$$\frac{(1-c^n)(1-c^{n+1})(1-c^{n+2})}{(1-c)(1-c^2)(1-c^3)}.$$

5. A circle is described about a triangle $ABC$, and from any point $D$ lines $DB$, $DC$ are drawn cutting the circle in two points $P$ and $Q$ whose pedal lines intersect in $S$. Prove that the angle $S$ is equal to the difference between the angles $A$ and $D$.

## Paper XXXI.

1. In a bag there is a number of tickets marked with the natural numbers from 1 to $n^2 + 1$. Every number is marked on each of $r$ tickets, and every square number $m^2$ confers a prize of $m$ shillings. A person can draw one ticket from the bag. Shew that the value of his expectation is $\frac{n^2(n+1)^2}{2(n^2+1)(n^2+2)}$.

2. If $(1 + x + x^2)^n = P_0 + P_1x + P_2x^2 + \ldots + P_rx^r + \ldots$

prove (1) $P_n = P_0^2 - P_1^2 + P_2^2 - \ldots$

$$(2)\ P_n = \frac{1}{(n-1)!}\left\{\frac{(2n-1)!}{n!} - n\cdot\frac{(2n-4)!}{(n-3)!} + \frac{n(n-1)}{2!}\cdot\frac{(2n-7)!}{(n-6)!} - \ldots\right\}.$$

5. $A$, $B$, $C$, $D$ are four points not in one plane. If $AB$ is perpendicular to $CD$, and $AC$ is perpendicular to $BD$, then will $AD$ be perpendicular to $BC$.

6. $TP$, $TQ$ are tangents to a parabola whose focus is $S$. $LM$, a third tangent cuts them in $L$ and $M$. Prove that the triangles $SPL$, $STM$ are similar.

Hence shew that $TL : LP :: QM : MT$.

## Paper XXXII.

1. Of three events it is 2 to 1 against the first and second happening, 3 to 2 against the second and third, and 9 to 1 against the first and third. Shew that the odds against all three happening are $5\sqrt{3} - 1$ to 1.

2. $O$ is the centre of gravity of a triangle. $AO$, $BO$, $CO$ are produced to points $D$, $E$, $F$ such that $AD = l\,.\,AO$, $BE = m\,.\,BO$, $CF = n\,.\,CO$. Find the values of $l$, $m$, $n$ so that the sides of the triangle $DEF$ may pass through the points $A$, $B$, $C$.

5. $A$, $B$, $C$, $D$ are four points in space. $AB$, $AC$ are divided in $E$, $F$ so that $AE : EB :: AF : FC$. $DB$, $DC$ are divided in $G$, $H$ so that $DG : GB :: DH : HC$. Shew that the lines $GF$ and $HE$ will intersect.

## PAPER XXXIII.

1. A river flows from $P$ to $Q$, a distance of 12 miles, at a uniform rate. $B$ starts at 12 o'clock from $Q$ to row to $P$, and $A$ starts at 5 minutes past 12 to row from $Q$ to $P$ and back again. $A$ overtakes $B$ a mile from $Q$; he rows on to $P$, and at once turning back meets $B$ two miles below $A$. $A$ reaches $Q$ 35 minutes after $B$ reaches $P$. Find the times at which $A$ passed $B$, and the rate of the stream.

5. Any point $P$ is taken on a given segment of a circle described on a line $AB$, and perpendiculars $AG$ and $BH$ are let fall on $BP$ and $AP$ respectively. Prove that $GH$ touches a fixed circle.

## PAPER XXXIV.

1. A policeman walks round his beat uniformly during his hours of duty. Shew that the chance of my meeting him, if I walk *in the opposite direction* down a street, which is $\frac{1}{n}$th of his beat, at a rate $m$ times his, is $\frac{1+m}{mn}$, where $m > 1$.

Also solve the problem when the condition in italics is removed.

2. If $b.\frac{y}{z}+c.\frac{z}{y}=a$, $c.\frac{z}{x}+a.\frac{x}{z}=b$, $a.\frac{x}{y}+b.\frac{y}{x}=c$ then will $x^{-3}+y^{-3}+z^{-3}+x^{-1}y^{-1}z^{-1}=0$, $a^3x^3+b^3y^3+c^3z^3+abcxyz=0$, $a^3+b^3+c^3=5abc$.

4. Prove that the distance between the centre of the inscribed circle and the orthocentre of a triangle is

$$2R\,(\text{vers}\,A\,\text{vers}\,B\,\text{vers}\,C-\cos A\cos B\cos C)^{\frac{1}{2}},$$

where $R$ is the radius of the circum-circle.

## PAPER XXXV.

2. Solve the equations :—

(1) $\dfrac{x-\sqrt{x^2-1}}{\sqrt{x+\sqrt{x^2-1}}}=\sqrt[4]{x^2-1}\left\{\sqrt{x^2+x}-\sqrt{x^2-x}\right\}$.

(2) $x(y+z)^2 = 1 + a^3$; $x + y = \frac{2}{2} + x$; $yz = \frac{3}{16}$.

(3) $\left.\begin{array}{c} a(y-z) + b(z-x) + c(x-y) = 0 \\ (x-y)(y-z)(z-x) = d^3 \\ x + y + z = e \end{array}\right\}$

3. Prove that in a triangle where $a < c$,

$$\frac{\cos nA}{b^n} = \frac{1}{c^n}\left\{1 + n\cdot\frac{a}{c}\cos B + \frac{n(n+1)}{2\,1}\frac{a^2}{c^2}\cos 2B + \frac{n(n+1)(n+2)}{3\,1}\frac{a^3}{c^3}\cos 3B + \ldots\right\}.$$

## Paper XLIII.

6. An ellipse and hyperbola are described so that the foci of each are at the extremities of the transverse axis of the other. Prove that the tangents at their points of intersection meet the conjugate axis in points equidistant from the centre.

## Paper XLIV.

1. If $x$, $y$, $z$ are in G.P. when $\theta_1$ is subtracted from each; and $z$, $y$, $x$ are in G.P. when $\theta_2$ is subtracted from each; and $x$, $z$, $y$ are in G.P. when $\theta_3$ is subtracted from each; prove that

$$\frac{1}{\theta_1 - x} + \frac{1}{\theta_2 - y} + \frac{1}{\theta_3 - z} = 0.$$

4. If $TA$, $TB$ be tangents meeting a circle in $A$ and $B$, and $TCQD$ be any chord meeting the circle in $C$ and $D$, whilst $Q$ is the middle point of the chord $CD$, shew that $TQ$ bisects the angle $AQB$, and the length of $TQ$ varies as the sum of the lengths of $AQ$ and $BQ$.

5. $T$ is any point on the tangent to a parabola at $Q$. Prove that the tangent at $T$ to the circle round $TQS$ touches the parabola.

## Paper LI.

1. Prove that the value of the expression

$$\sqrt{a^2\cos^2\phi + b^2\sin^2\phi} + \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}$$

lies between $a + b$ and $\sqrt{2a^2 + 2b^2}$.

Prove also that

$$\frac{1}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} + \frac{1}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\phi}}$$

lies between $\frac{1}{a} + \frac{1}{b}$ and $\frac{4}{\sqrt{2a^2 + 2b^2}}$.

5. $ABC$ is a triangle inscribed in a conic whose centre is $O$, and $Oa$, $Ob$. $Oc$ are drawn to the middle points of the chords. From any point $P$ on the conic, $Pa$, $P\beta$, $P\gamma$ are drawn parallel to $Oa$, $Ob$, $Oc$ to meet the sides in $a$, $\beta$, $\gamma$. Prove that the points $a$, $\beta$, $\gamma$ are collinear.

## Paper LV.

3. The sides of a triangle are in A.P., and its area is to that of an equilateral triangle of the same perimeter :: 3 : 5.
Shew that the greatest angle is 120°.

5. $PAQ$, $PBC$ are two semi-circles which touch internally at $P$, $PQO$ being the common diameter. Through $P$ draw a secant $PAB$ such that the area of the triangle $ABC$ may be a max., and shew that for this position of the secant the area of the triangle $QAB$ is also a max.

## Paper LVII.

2. In the continued fraction

$$\frac{1}{(1-x)+}\ \frac{x}{(1-x^3)+}\ \frac{x^3}{(1-x^5)+}\ \frac{x^5}{(1-x^7)+}\ \cdots$$

shew that the $n^{\text{th}}$ convergent is $\frac{\sigma_n}{\sigma_n - 1}$,

where $\sigma_n = x^{-1^2} - x^{-2^2} + x^{-3^2} - x^{-4^2} + \ldots + (-1)^{n-1} \cdot x^{-n^2}$.

7. Prove geometrically that if a line be drawn through a focus of a central conic making a constant angle with a tangent, the locus of the point of intersection is a circle.

## Paper LIX.

5. $ABC$ is a triangle, $O$ any point, in the same plane or not; $P$, $Q$, $R$ points in $OA$, $OB$, $OC$. $BR$, $CQ$ intersect in $L$; $CP$, $AR$ in $M$; $AQ$, $BP$ in $N$. $OL$, $OM$, $ON$ cut $BC$, $CA$, $AB$ in $D$, $E$, $F$. Prove that $AD$, $BE$, $CF$ are concurrent.

6. A semicircular piece of paper is folded over so that a particular point $P$ on the bounding diameter lies on the circular boundary. Shew that the crease-line always touches a fixed conic.

7. A straight line of given length moves so that its extremities always lie (1) on a fixed ellipse, (2) on a fixed parabola. Find the locus of its middle point in the two cases.

## Paper LX.

2. Shew how to find $n$ if the sum of $n$ terms of the series

$$1 + 5 + 9 + 13 + \ldots$$

be a perfect square; and find the first two values of $n$ greater than unity.

4. From a point $A$ on the outer of two concentric circles tangents $AP, AQ$ are drawn to the inner. $AP, QP$ meet the outer again in $T, R$. Prove that

$$RP : RQ :: RT^2 : RA^2.$$

## Paper LXI.

4. The sum of the reciprocals of the distances of a fixed point from tangents to a circle at the extremities of any chord through the point is constant.

## Paper LXVII.

7. In the system of pulleys in which each string is attached to a bar supporting the weight, find at what point of the bar the weight must be attached if there are two movable pulleys.

Also shew that if the weight be then doubled, it will descend with acceleration $= \frac{g}{15}$.

## Paper LXX.

2. If $x_2x_3 + y_2y_3 = x_3x_1 + y_3y_1 = x_1x_2 + y_1y_2 = 1$,

and $d_1 = x_2y_3 - x_3y_2$, $d_2 = x_3y_1 - x_1y_3$, $d_3 = x_1y_2 - x_2y_1$,

shew that $d_1 + d_2 + d_3 = d_1d_2d_3$.

## PAPER LXXV.

6. $P$ is any point on a conic circumscribing the triangle $ABC$, and the diameters which bisect the chords parallel to $PA$, $PB$, $PC$ meet the tangents at $A$, $B$, $C$ in the points $D$, $E$, $F$ respectively. Shew that $D$, $E$, $F$ lie on the polar of $P$.

## PAPER LXXVI.

7. A particle of elasticity $e$ is projected from a point in the wall of a square room in a direction whose projection on the floor makes an angle $\theta$ with the wall. Shew that if the particle after striking each wall in succession returns to the point of projection, then

$$e(\mu + 1) \cot\theta = e\mu + 1,$$

$\mu : 1$ being the ratio in which a horizontal line in the side of the wall is divided by the point of projection.

## PAPER LXXVII.

5. If in a rough inclined plane the ratio of the greatest force to the least force which, acting parallel to the plane, will just support a given weight on the plane be equal to the ratio of the weight to the pressure on the plane, prove that the coefficient of friction is $\tan a \cdot \tan^2 \frac{a}{2}$, where $a$ is the inclination of the plane.

## PAPER LXXVIII.

7. A projectile is discharged with velocity $v$ at an elevation $\alpha$, and $n$ seconds afterwards a second one is discharged after it so as to strike it. If $v'$, $\alpha'$ be its velocity and elevation, prove that

$$2vv' \sin(\alpha - \alpha') = (v \cos\alpha + v' \cos\alpha')\, gn.$$

## PAPER LXXIX.

3. Prove that

$$(1)\quad 1 = \tan\frac{\pi}{2^{n+1}} \left\{ \tan\frac{\pi}{2^{n+1}} + 2\tan\frac{\pi}{2^n} + \ldots + 2^{n-1}\tan\frac{\pi}{2^3} + 2^{n-1} \right\}.$$

$$(2)\quad 2 = \frac{1}{\cos^2\frac{\theta}{2}} + \frac{1}{2}\frac{\cos\theta}{\cos^2\frac{\theta}{2}} + \frac{1}{2^2}\frac{\cos\theta\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}\cos^2\frac{\theta}{2^2}} + \frac{1}{2^3}\frac{\cos\theta\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}}{\cos^2\frac{\theta}{2}\cos^2\frac{\theta}{2^2}\cos^2\frac{\theta}{2^3}} + \ldots$$

and sum to $n$ terms the series

$$\sec^2\theta + 2^2\sec^2 2\theta + 2^4\sec^2 2^2\theta + \ldots + 2^{2n-2}\sec^2 2^{n-1}\theta.$$

4. On the sides of a triangle as bases are described externally three similar isosceles triangles. Prove geometrically that the lines joining the vertices of these triangles with the opposite vertices of the given triangle are concurrent.

5. Shew that the equation of the envelope of a circle described upon a chord of the circle $(x - a)^2 + y^2 = c^2$ passing through the origin as diameter is

$$(x^2 + y^2 + a^2 - c^2)(x^2 - 2ax + y^2 + a^2 - c^2) = a^2y^2.$$

Prove also that the maximum distance of a point on the envelope from the centre of the given circle is $c\sqrt{2}$.

## Paper LXXX.

7. A tennis ball is served from a height of 8 feet. It just touches the net at a point where the net is 3 ft. 3 in. high, and hits the service line, 21 feet from the net. The horizontal distance of the server from the foot of the net is 39 feet. Prove that the angle which the direction of projection makes with the horizontal is $\tan^{-1}\frac{137}{1365}$; and that the horizontal velocity of the ball is about 160 feet per second, the plane of projection being perpendicular to the plane of the net.

## Paper LXXXV.

4. $S$ and $H$ are the foci of a hyperbola, and $PT$, the tangent at $P$, cuts an asymptote in $T$. Prove that the angle $STP = PHT$.

## PAPER LXXXVIII.

7. One end of a string is fixed to a beam, from which it passes downwards and under a movable pulley of weight $P$, then over a fixed pulley, then under a second movable pulley of the same weight, and then the other end is attached to the first movable pulley. A weight $W$ is attached to the second movable pulley, and all the straight portions of the string are vertical.

Prove that there will be equilibrium if $W = P$.

Also, if $W > P$, the downward acceleration of $W$ will be

$$\frac{W - P}{W + 5P} g.$$

## PAPER XCII.

7. If on a rectangular billiard table whose sides are $a$, $b$, a ball describe a rectangle whose sides are $c$, $d$, prove that the coefficient of elasticity between the ball and the sides of the table is

$$\left(\frac{ad - bc}{bd - ac}\right)^2 \text{ or } \left(\frac{bd - ac}{ad - bc}\right)^2.$$

## PAPER XCIII.

5. The envelope of a perpendicular drawn to a normal to a parabola at the point where the normal cuts the axis is a parabola. Prove also that the focal vector of the point of the parabola at which the normal is drawn meets the envelope at the point where the perpendicular touches it.

6. Shew that if $x_1^3 + y_1^3 = x_2^3 + y_2^3 = x_3^3 + y_3^3 = a^3$,

and $$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0,$$

then $$x_1x_2x_3 + y_1y_2y_3 = a^3.$$

A straight line cuts in 3 real points the curve $x^3 + y^3 = a^3$. Shew that their centroid, if it lie on either axis of coordinates, will be at the origin.

## PAPER XCIV.

1. Prove that

$3 \cdot 81^{n+1} + (16n - 54)\, 9^{n+1} - 320n^2 - 144n + 243$ is a multiple of $2^{12}$.

5. The diameter $d$ of a circle is divided into $2n$ equal parts, and straight lines are drawn from any point in the circumference to each

point of division. If $a_1, a_2 \ldots a_{2n-1}$ be the lengths of the lines so drawn, prove that in the limit, when the number of parts is increased indefinitely,

$$a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a^2{}_{2n-1} = \frac{d^2}{2}.$$

## Paper XCV.

7. From a point at a distance $d$ from a plane whose inclination is $\beta$, two particles are projected simultaneously with velocities $u$ and $v$ in two different directions parallel to the plane and at right angles to each other. Prove that they will strike the plane simultaneously at points $A$ and $B$ such that

$$AB^2 = \frac{2d}{g}(u^2 + v^2)\sec\beta.$$

## Paper XCVI.

1. If $p = a + \frac{a^2}{a}$, $q = b + \frac{x^2}{b}$, $r = c + \frac{x^2}{c}$, prove that

$$\frac{1}{a}\left\{1 - \frac{q-r}{b-c}\right\} = \frac{1}{b}\left\{1 - \frac{r-p}{c-a}\right\} = \frac{1}{c}\left\{1 - \frac{p-q}{a-b}\right\};$$

and eliminate $x, y, z$ from the equations

$$p = x - \frac{yz}{x},\ q = y - \frac{zx}{y},\ r = z - \frac{xy}{z},\ \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0.$$

2. If $a, b, c, d$ be the sides of a quadrilateral taken in order, and $\phi$ the angle between the diagonals, shew that the area of the quadrilateral is

$$\tfrac{1}{4}(a^2 - b^2 + c^2 - d^2)\tan\phi.$$

## Paper XCVII.

1. Prove that the max. and min. values of

$$x^3 + 3px^2 + 3qx + r$$

are

$$2p^3 - 3pq + r \pm 2(p^2 - q)^{\frac{3}{2}}.$$

4. If tangents be drawn to a fixed circle from any point on another circle, the envelope of the chord of contact is a conic.

7. If a particle of mass $m$ fall down a cycloid under the action of gravity starting from the cusp, prove that the pressure of the particle upon the cycloid at any point is $2mg\cos\psi$, where $\psi$ is the inclination to the horizon of the tangent to the cycloid at the point; also shew that the resultant acceleration $= g$.

## PAPER XCVIII.

6. Two equal uniform ladders, each of length $l$ and weight $w$, are freely jointed at $A$ and are connected by a rope $PQ$. A man whose weight is $W$ goes $b$ feet up one of the ladders. If the ground be smooth, prove that the tension of the rope

$$= \frac{Wb + wl}{2a} \cdot \frac{c}{\sqrt{a^2 - c^2}},$$

where $2c$ is the length of the rope in feet, and $a = AP = AQ$.

## PAPER XCIX.

4. Two triangles $BAC$, $BA'C$ are inscribed in a circle on the common base $BC$, and the pedal lines of the triangles $BAC$, $BA'C$ are formed with regard to the points $A'$ and $A$ respectively. Shew that these two lines and the nine points' circles of the two triangles intersect in the same point.

## PAPER C.

2. Eliminate $\theta$, having given

$$x\cos(\theta - \alpha) + y\cos\theta = 2a\sin(\theta + \gamma)\cos\theta\cos(\theta - \alpha),$$

$$x\sin(\theta - \alpha) + y\sin\theta = 2a\left\{\sin(\theta + \gamma)\sin\theta\cos(\theta - \alpha) - \cos\beta\cos\theta\right\},$$

$$\alpha + \beta + \gamma = \pi.$$

# SOLUTIONS OF ADDITIONAL PROBLEMS.

## PAPER VI.

7. Let $CD$ be the diameter of the fixed ellipse conjugate to $CP$, and on the normal at $P$ take $PH = PH' = CD$. Then by Salm. *Con.* Art. 181 (*a*), the angle $HCH'$ is bisected by one of the axes of the fixed ellipse. $\therefore$ the other axis touches an ellipse whose foci are $H$ and $H'$, and which passes through $C$. Also $CH = a - b$, $CH' = a + b$.

## PAPER VII.

5. The angle $\beta a\gamma = \beta aA' + A'a\gamma = BB'A' + A'C'C = BAA' + CAA' = BAC$.

The angle $a\beta\gamma = B'\beta\gamma - B'\beta a = \pi - B'C'C - B'A'a$

$$= \pi - B'BC - (\pi - B'A'A)$$

$$= B'BA - B'BC = ABC.$$

$\therefore$ the angle $\beta\gamma a = BCA$. $\therefore a\beta\gamma$ is similar to $ABC$.

## PAPER XI.

6. The condition that the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

should represent two straight lines is

$$abc - af^2 - bg^2 - ch^2 + 2fgh = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The asymptotes of the given equation are

$$11x^2 + 24xy + 4y^2 - 2x + 16y + 11 + \lambda = 0 \quad . \quad . \quad (2)$$

where $\lambda$ is such a quantity as will make (2) represent two straight lines.

$\therefore$ by substitution in (1) we find $\lambda = -20$. $\therefore$ the required equation is

$$11x^2 + 24xy + 4y^2 - 2x + 16y - 9 = 0,$$

or
$$(11x + 2y + 9)(x + 2y - 1) = 0.$$

The centre is the intersection of the asymptotes, and is given by

$$11x + 2y + 9 = 0\ ;\quad x + 2y - 1 = 0.\quad \therefore x = -1,\ y = 1.$$

Transferring to parallel axes through the centre, the equation becomes

$$11x^2 + 24xy + 4y^2 = 2.$$

The equation of a concentric circle is

$$x^2 + y^2 = r^2.$$

The equation of the common chords is

$$\left(\frac{11}{2} - \frac{1}{r^2}\right)x^2 + 12xy + \left(2 - \frac{1}{r^2}\right)y^2 = 0.$$

These coincide and are the axes when

$$\left(\frac{11}{2} - \frac{1}{r^2}\right)\left(2 - \frac{1}{r^2}\right) = 36.\quad \therefore r^2 = \tfrac{11}{10} \text{ or } -\tfrac{7}{5}.$$

$\therefore$ the semi-axes are $\sqrt{\frac{11}{10}},\ \sqrt{-\frac{7}{5}}$.

$\therefore$ the curve is a hyperbola, of which one branch lies entirely between the positive axis of $x$ and the negative axis of $y$, the centre and the other branch being in the opposite quadrant.

The equation of a pair of tangents from $x'y'$ to the hyperbola is

$$(11x^2 + 24xy + 4y^2 - 2x + 16y + 11)(11x'^2 + 24x'y' + 4y'^2 - 2x' + 16y' + 11)$$
$$= \left\{11xx' + 12(xy' + x'y) + 4yy' - (x + x') - 8(y + y') + 11\right\}^2.$$

These two tangents will be at right angles if the sum of the coefficients of $x^2$ and $y^2$ is zero.

$$\therefore\ 15(11x'^2 + 24x'y' + 4y^2 - 2x + 16y + 11)$$
$$= (11x + 12y - 1)^2 + (12x + 4y - 8)^2.$$

$\therefore$ the equation to the locus is

$$x^2 + y^2 + 2x - 2y = 1.$$

## Paper XII.

7. Bisect $AB$ in $O$. Then $O$ is the centre of the circle. Produce $PO$ to meet the circumference in $Q$. Complete the parallelogram $CPQR$. The resultant of $PA$ and $PB = 2PO = PQ$. The resultant of $PQ$ and $PC = PR$. Now $CP = PQ$ = diameter of given circle = const. $\therefore$ locus of $R$ is a circle, centre $C$, radius = diameter of given circle.

## Paper XVII.

7. Let $ABCDEF$ be the hexagon, and let the forces 1, 2, . . . act along $AB$, $BC$ . . . Let $O$ be the centre, and $AB = a$. The distance of $O$ from any side $= a\frac{\sqrt{3}}{2}$. If $R$ denote the resultant, $p$ the perpendicular from $O$ on its direction,

$$pR = a\frac{\sqrt{3}}{2}(1 + 2 + 3 + 4 + 5 + 6) = a \cdot 42\sqrt{3}.$$

Resolving parallel and perpendicular to $AF$, we have

$$R^2 = (6 - 3 + 1\tfrac{1}{2} - 2\tfrac{1}{2} - 4\tfrac{1}{6} + 5\tfrac{1}{2})^2 + \left\{\frac{\sqrt{3}}{2}(1 + 2 - 4 - 5)\right\}^2$$

$$= 9 + 3 \times 9 = 36. \quad \therefore R = 6.$$

$$\therefore p = \frac{a\sqrt{3}}{2} \cdot 3\tfrac{1}{2} = 3\tfrac{1}{2} \times \text{radius of inscribed circle.}$$

## Paper XVIII.

1. Let $AB$, $BC$ be the two roads, and let $D$ be one of the points required.

Let $AB = a$, $BD = x$. Then $AD = \sqrt{x^2 + a^2}$.

$\therefore 7\sqrt{x^2 + a^2} = 5(x + a)$, $\therefore 24x^2 - 50ax + 24^2 = 0$, $\therefore x = \frac{4a}{3}$ or $\frac{3a}{4}$.

$\therefore$ there are two positions of $D$.

On $BC$ take a point $E$, and let $BE = y$. Then $AE = \sqrt{y^2 + a^2}$.

Then the time across country is quicker than the time by road if

$$7\sqrt{y^2 + a^2} > 5(a + y),$$

*i.e.* if $24(y^2 + a^2) - 50ay$ is positive,

*i.e.* if $24\left(y - \frac{4a}{3}\right)\left(y - \frac{3a}{4}\right)$ is positive,

*i.e.* if $y$ does not lie between $\frac{4a}{3}$ and $\frac{3a}{4}$.

2. Let $n$ be any number. Then by elementary algebra we find

$$n(n + 2)(n + 4)(n + 6) + 16 = (n^2 + 6n + 4)^2.$$

First let $n$ be odd, $= 2p + 1$,

$$\therefore N = n^2 + 6n + 4 = 4p^2 + 16p + 11.$$

Let $p = 5q + r$,

$$\therefore N = 100q^2 + 120qr + 4r^2 + 16r + 11$$

$= M + 4r^2 + 16r + 11$, where $M$ is a multiple of 10.

If $r = 0$, $N = M + 11$; $r = 1$, $N = M + 31$; $r = 2$, $N = M + 59$;
$r = 3$, $N = M + 95$; $r = 4$, $N = M + 139$.

$\therefore$ when $r = 0, 1, 2, 4$, the last figure of $N^2$ is 1.

when $r = 3$, the last figure of $N^2$ is 5.

Next let $n$ be even, and $= 2p$. Then as before we get

$$N = M + 4r^2 + 12r + 4.$$

If $r = 0$, $N = M + 4$; $r = 1$, $N = M + 20$; $r = 2$, $N = M + 44$;
$r = 3$, $N = M + 76$; $r = 4$, $N = M + 116$.

$\therefore$ when $r = 0, 2, 3, 4$, the last figure of $N^2$ is 6.

when $r = 1$, the last figure of $N^2$ is 0.

3. Let $P$ be the point, $O$ the circumcentre. Join $O$ and $P$ to the points $A$, $B$, $C$.

Let $COP = \theta$.

Then
$$l^2 = R^2 + d^2 - 2Rd\cos(2B - \theta),$$
$$m^2 = R^2 + d^2 - 2Rd\cos(2A - \theta),$$
$$n^2 = R^2 + d^2 - 2Rd\cos\theta.$$
$$\therefore l^2\sin 2A + m^2\sin 2B + n^2\sin 2C$$

$= (R^2 + d^2)(\sin 2A + \sin 2B + \sin 2C)$

$- 2Rd\{\sin 2A\cos(2B - \theta) + \sin 2B\cos(2A - \theta) + \sin 2C\cos\theta\}$

$= 4(R^2 + d^2)\sin A\sin B\sin C$

$- 2Rd\{\cos\theta(\sin 2A\cos 2B + \sin 2B\cos 2A + \sin 2C)\}$

$= 4(R^2 + d^2)\sin A\sin B\sin C.$

7. This is the same as shewing that the locus of the centre of the conic through the angular points and the orthocentre of a triangle is the nine points' circle, for the centre of the inscribed circle is the orthocentre of the triangle formed by the centres of the escribed circles, and the nine points' circle of the latter triangle is the circumcircle of the original triangle.

Let $ABC$ be a triangle, $O$ its orthocentre. Produce $AO$ to meet $BC$ in $D$.

Take $DC$, $DA$ as axes of $x$ and $y$, and let

$$DC = h, \quad DB = -k, \quad ABC = \theta.$$

Then $$AD = k\tan\theta, \quad OD = h\cot\theta.$$

The equation of a conic through $A$, $B$, $C$ and $O$ is

$$x^2 + 2\mu xy - y^2 - (h-k)x + (h\cot\theta + k\tan\theta)y - hk = 0$$

where $\mu$ is a variable. The centre is given by the equations

$$x + \mu y - \frac{h-k}{2} = 0\,; \quad \mu x - y + \frac{h\cot\theta + k\tan\theta}{2} = 0.$$

$\therefore$ eliminating $\mu$, the locus of the centre is

$$2(x^2+y^2) - (h-k)x + (h\cot\theta + k\tan\theta)y = 0.$$

This is the equation to a circle through $D$ and the middle points of $BC$ and $AO$. $\therefore$ it represents the nine points' circle.

## Paper XIX.

2. (1). $$x^4 + a^4 = 4ax(x^2+a^2),$$

$$\therefore 6a^2x^2 = x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4 = (x-a)^4,$$

$$\therefore (x-a)^2 = \pm\sqrt{3}.ax, \ \therefore x^2 - (2 \pm \sqrt{3})ax + a^2 = 0,$$

$$\therefore 2x = a\left\{2 \pm \sqrt{3} \pm 3^{\frac{1}{4}}(\sqrt{3}\pm 4)^{\frac{1}{2}}\right\}.$$

(2). $$\frac{x}{a} + \frac{b}{y} = 1 - \frac{c}{z} = \frac{a}{x} + \frac{y}{b}. \ \therefore (xy+ab)(ay-bx) = 0.$$

$\therefore$ either $xy + ab = 0$. $\therefore \dfrac{x}{a} + \dfrac{b}{y} = 0$. $\therefore z = c$, and $\dfrac{x}{a} + \dfrac{y}{b} = 0$,

$$\left.\begin{array}{l} \therefore ay + bx = 0 \\ ay - bx = \pm 2ab \end{array}\right\} \therefore 4a^2b^2 = (ay+bx)^2 - 4abxy = (ay-bx)^2$$

$$\therefore y = \pm b, \ x = \mp a, \ z = c.$$

or $ay - bx = 0$, $\therefore \dfrac{a}{x} = \dfrac{b}{y}$, $\therefore 1 - \dfrac{z}{c} = \dfrac{2a}{x}$, $\therefore z = \dfrac{c(x-2a)}{x}$,

and $$\frac{x}{a} + \frac{a}{x} = 1 - \frac{c}{z} = -\frac{2a}{x-2a}, \ \therefore (x^2+a^2)(x-2a) = -2a^2x,$$

$$\therefore (x-a)(x^2 - ax + 2a^2) = 0,$$

$\therefore x = a$, or $\frac{a}{2}(1 \pm \sqrt{-7})$. $\therefore y = \frac{b}{a}x = b$, or $\frac{b}{2}(1 \pm \sqrt{-7})$,

$$z = c\left(1 - \frac{2a}{x}\right) = -c \text{ or } \frac{c}{2}(1 \mp \sqrt{-7}).$$

4. $R\sin A = \frac{a}{2}$, $\sin 2A = 2\sin A\cos A = \frac{4S}{bc} \cdot \frac{b^2 + c^2 - a^2}{2bc}$,

$$\therefore R^2\sin^3 A\cos(B - C) = R^2\sin^2 A \,.\, \sin(B + C)\cos(B - C)$$

$$= \frac{a^2}{8}(\sin 2B + \sin 2C)$$

$$= \frac{a^2}{4} \cdot S\left(\frac{c^2 + a^2 - b^2}{c^2a^2} + \frac{a^2 + b^2 - c^2}{a^2b^2}\right)$$

$$= \frac{S}{4a^2b^2c^2}(2a^2b^2c^2 + a^4b^2 - a^2b^4 + a^4c^2 - a^2c^4).$$

$\therefore$ the given expression on the right

$$= \frac{2}{3} \cdot \frac{6Sa^2b^2c^2}{4a^2b^2c^2} = S.$$

5. Let $BAC$ be a triangle. Bisect the angle at $A$ by $AD$, and in $BD$ take any point $F$. Join $AF$, and at the point $A$ make the angle $DAG = DAF$, $G$ being on the circum-circle.

Then the angle $BAF = CAG$, and $ABF = ACG$ in the same segment, $\therefore$ the triangles $ABF$, $AGC$ are similar.

$$\therefore BA : AF :: AG : AC. \quad \therefore AB \,.\, AC = AF \,.\, AG.$$

## Paper XX.

3. Let $AB = a$, $BC = b$, $CD = c$, $DA = d$.

Then $DB^2 = a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C$,

$$\therefore a^2 + d^2 - b^2 - c^2 = 2ad\cos A - 2bc\cos C,$$

$$\therefore (a + d)^2 - (b - c)^2 = 4ad\cos^2\frac{A}{2} + 4bc\sin^2\frac{C}{2},$$

and $$(a - d)^2 - (b + c)^2 = -4ad\sin^2\frac{A}{2} - 4bc\cos^2\frac{C}{2},$$

$$\therefore 16(s - a)(s - b)(s - c)(s - d)$$

$$= 16\left(bc\sin^2\frac{C}{2} + ad\cos^2\frac{A}{2}\right)\left(bc\cos^2\frac{C}{2} + ad\sin^2\frac{A}{2}\right)$$

$$= 16\left\{\left(bc\sin\frac{C}{2}\cos\frac{C}{2}+ad\sin\frac{A}{2}\cos\frac{A}{2}\right)^2+abcd\left(\cos\frac{C}{2}\cos\frac{A}{2}-\sin\frac{C}{2}\sin\frac{A}{2}\right)^2\right\}$$

$$= 4(bc\sin C + ad\sin A)^2 + 16\,abcd\cos^2\frac{A+C}{2}$$

$$= (\text{area of quadrilateral})^2 + 16\,abcd\cos^2\frac{A+C}{2}.$$

$$\therefore \text{ if area } = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

we must have $\cos\frac{A+C}{2} = 0$, $\therefore A + C = 180°$. $\therefore$ the quadrilateral can be inscribed in a circle.

4. Let $OE$, $OF$ be the tangents at $O$ to the circles round $AOC$, $BOD$ respectively. Then the angle $COE = OAC$, and $COF = OBD$, because they are in alternate segments. $\therefore FOE = OAC \sim OBD = APB$.

Now let $OP$ produced meet the circle round $AOC$ in $H$.

Then $OP.PH = AP.PC = BP.PD$, $\therefore H$ is also a point on the circle round $BOD$.

5. Let $W$ be the weight of the rod, $R$, $R'$ the total resistances at $A$ and $B$. Let $C$ be the centre of the wire. Join $CA$, $CB$.

Since the rod is in equilibrium, the directions of $W$, $R$, $R'$ meet in a point. Let this be $O$. Since the equilibrium is limiting, the angle $OAC = OBC$. $\therefore O$, $A$, $B$, $C$ are points on a circle. This gives the required result.

6. Produce $PS$ to meet the major axis in $N$, and draw the ordinate $QM$.

Then $SN = QM$, and $PN^2 = AN.NA'$,

$\therefore SN^2 : AN.NA' :: QM^2 : PN^2 :: OQ^2 : OP^2 :: b^2 : a^2$.

$\therefore S$ is on the ellipse.

Produce $RS$ to meet the major axis in $G$.

Then $OG : QS\,(= MN) :: OR : OP$.

And $MN : ON :: PQ : OP$,

$\therefore OG : ON :: OR.PQ : OP^2$

$:: (a+b)(a-b) : a^2$

$:: a^2 - b^2 : a^2$.

$\therefore SG$ is the normal at $S$.

7. The abscissa of a point of intersection $(x', y')$ is given by

$$(x'-a)(x'-a') = (x'-b)(x'-b'),\ \therefore x'(a+a'-b-b') = aa'-bb',$$

$$\therefore x'-a = -\frac{(a-b)(a-b')}{a+a'-b-b'};\ x'-a' = -\frac{(a'-b)(a'-b')}{a+a'-b-b'}$$

$$\therefore y'^2 = -(x'-a)(x'-a') = -\frac{(a-b)(a-b')(a'-b)(a'-b')}{(a+a'-b-b')^2}.$$

The equations of the tangents at $(x', y')$ are

$$2(xx'+yy') - (a+a')(x+x') + 2aa' = 0,$$

$$2(xx'+yy') - (b+b')(x+x') + 2bb' = 0.$$

$$\therefore \tan\theta = \frac{\dfrac{a+a'-2x'}{y'} - \dfrac{b+b'-2x'}{y'}}{1+\dfrac{(a+a'-2x')(b+b'-2x')}{y'^2}}$$

$$= \frac{2y'(a+a'-b-b')}{(a-b')(b-a')+(a-b)(b'-a')},$$

$$\therefore \frac{4y'^2(a+a'-b-b')^2}{\sin^2\theta} = \frac{\{(a-b')(b-a')+(a-b)(b'-a')\}^2}{\cos^2\theta}$$

$$= 4y'^2(a+a'-b-b')^2 + \{(a-b')(b-a')+(a-b)(b'-a')\}^2$$

$$= -4(a-b)(a-b')(a'-b)(a'-b') + (a-b')^2(a'-b)^2$$
$$+ 2(a-b)(a-b')(a'-b)(a'-b') + (a-b)^2(a-b')^2$$

$$= \{(a-b')(a'-b) - (a-b)(a'-b')\}^2$$

$$= \{-(a-a')(b-b')\}^2$$

$$\therefore (a-a')^2(b-b')^2\sin^2\theta + 4(b'-a)(b-a)(b'-a')(b-a') = 0.$$

## Paper XXI.

1. The population 10 years ago $= 12100 - 11000 = 1100$,

$\therefore$ the numbers in three consecutive years were 1000, 1100, 1210, which are terms of a G.P. common ratio being $\frac{11}{10}$.

2. We have to prove that $5\sin^{-1}\frac{1}{\sqrt{50}} = \frac{\pi}{4} - 2\sin^{-1}\frac{3}{25\sqrt{10}}$.

Let $a = \sin^{-1}\frac{1}{\sqrt{50}}$, $\beta = \sin^{-1}\frac{3}{25\sqrt{10}}$,

$$\therefore \sin 5a = 5\sin a - 20\sin^3 a + 16\sin^5 a$$

$$= \frac{1}{\sqrt{50}}\left(5 - \frac{20}{50} + \frac{16}{50^2}\right) = \frac{2879}{5^5\sqrt{2}},$$

$$\sin\beta = \frac{3}{25\sqrt{10}}, \cos\beta = \frac{79}{25\sqrt{10}}, \quad \therefore \sin 2\beta = \frac{237}{5^5}$$

$$\therefore \cos^2 2\beta = \frac{1}{5^{10}}(5^{10} - 237^2) = \frac{1}{5^{10}} \times 9709456, \quad \therefore \cos 2\beta = \frac{3116}{5^5},$$

$$\therefore \sin\left(\frac{\pi}{4} - 2\beta\right) = \frac{1}{\sqrt{2}}(\cos 2\beta - \sin 2\beta)$$

$$= \frac{3116 - 237}{5^5\sqrt{2}} = \frac{2879}{5^5\sqrt{2}} = \sin 5a.$$

*This may also be solved by noticing that

$$\sin^{-1}\frac{1}{\sqrt{50}} = \tan^{-1}\tfrac{1}{7}, \sin^{-1}\frac{3}{25\sqrt{10}} = \tan^{-1}\tfrac{3}{79},$$

$$\therefore 5\tan^{-1}\tfrac{1}{7} + 2\tan^{-1}\tfrac{3}{79} = 3\tan^{-1}\tfrac{1}{7} + 2\tan^{-1}\tfrac{2}{11}$$

$$= \tan^{-1}\tfrac{1}{7} + 2\tan^{-1}\tfrac{1}{3}$$

$$= \tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3} = \frac{\pi}{4}, \text{ Euler's Series.}$$

NOTE.—From this question can be obtained the expansion of $\pi$ given in XXI. No. 1.

3. (1). Let $S = \sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{2^2}\sin 3\theta + \ldots$

$$C = \cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2^2}\cos 3\theta + \ldots$$

$$\therefore C + Si = e^{\theta i} + \frac{1}{2}e^{2\theta i} + \frac{1}{2^2}e^{3\theta i} + \ldots$$

$$= e^{\theta i}\left\{1 + \frac{e^{\theta i}}{2} + \left(\frac{e^{\theta i}}{2}\right)^2 + \ldots\right.$$

$$= e^{\theta i}\left(1 - \frac{e^{\theta i}}{2}\right)^{-1},$$

$$\therefore (2 - e^{\theta i})(C + Si) = 2e^{\theta i},$$

$$\therefore (2 - \cos\theta - i\sin\theta)(C + Si) = 2(\cos\theta + i\sin\theta),$$

$\therefore$ equating real and unreal parts,

$$\left.\begin{aligned}(2 - \cos\theta)\,C + S\,.\,\sin\theta &= 2\cos\theta\\ (2 - \cos\theta)\,S - C\,.\,\sin\theta &= 2\sin\theta\end{aligned}\right\}.$$

Eliminating $C$, we find $S = \dfrac{4\sin\theta}{5 - \cos\theta}$.

(2). Let $S = \sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta \ldots$

$$C = \cos\theta - \tfrac{1}{3}\cos 3\theta + \tfrac{1}{5}\cos 5\theta \ldots$$

$$\therefore C + Si = e^{\theta i} - \tfrac{1}{3}e^{3\theta i} + \tfrac{1}{5}e^{5\theta i}$$

$$= \frac{1}{i}\left\{(ie^{\theta i}) + \tfrac{1}{3}(ie^{\theta i})^3 + \tfrac{1}{5}(ie^{\theta i})^5 + \ldots\right\}$$

$$= \frac{1}{2i}\log\frac{1 + ie^{\theta i}}{1 - ie^{\theta i}}.$$

$$\therefore 2Ci - 2S = \log\frac{1 + ie^{\theta i}}{1 - ie^{\theta i}},$$

$$\therefore e^{2Ci}\,.\,e^{-2S} = \frac{1 + ie^{\theta i}}{1 - ie^{\theta i}}.$$

$\therefore$ $\therefore$ $(1 + \sin\theta - i\cos\theta)(\cos 2C + i\sin 2C)\,.\,e^{-2S} = 1 - \sin\theta + i\cos\theta,$

$\therefore$ equating real and unreal parts,

$$\left.\begin{aligned}(1 + \sin\theta)\cos 2C + \cos\theta\sin 2C &= (1 - \sin\theta)e^{2S}\\ (1 + \sin\theta)\sin 2C - \cos\theta\cos 2C &= \cos\theta\,.\,e^{2S}\end{aligned}\right\}.$$

If we square and add these equations, we shall eliminate $C$.

$$\therefore 2e^{4S}(1 - \sin\theta) = 2(1 + \sin\theta), \quad \therefore S = \tfrac{1}{4}\log\frac{1 + \sin\theta}{1 - \sin\theta}.$$

(3). Let $S = \sin\theta + \dfrac{1}{4}\sin 3\theta + \dfrac{1}{4^2}\sin 5\theta + \ldots$

$$C = \cos\theta + \frac{1}{4}\cos 3\theta + \frac{1}{4^2}\cos 5\theta + \ldots$$

$$\therefore C + Si = e^{\theta i}\left\{1 + \frac{e^{2\theta i}}{4} + \left(\frac{e^{2\theta i}}{4}\right)^2 + \dots\right\}$$

$$= e^{\theta i}\left(1 - \frac{e^{2\theta i}}{4}\right)^{-1}.$$

$\therefore$ proceeding as in (1) we find $S = \dfrac{20 \sin\theta}{9 + 16\sin^2\theta}$.

4. Since $AD$ bisects the angle $EDF$, we must bisect the angles at $D$, $E$, $F$, and draw lines through these points at right angles to the bisectors of the angles.

5. Let $Q$ be any point on the circumscribing circle, $QD$ perpendicular on $BC$, and let $P$ be the orthocentre. Bisect $PQ$ in $M$. Then since $P$ is a centre of similitude of the nine points' and circum-circles, $\therefore$ $M$ is on the nine points' circle. Join $MD$ cutting $AE$ in $E$. Join $AP$ and bisect it in $U$, and let $AP$ cut $BC$ in $X$. Then $M$, $U$, $X$ are on the N.P.C. And $MU = \frac{1}{2}QA$. $\therefore$ the angle in the segment of the circum-circle cut off by $QA$ = the angle in the segment of the nine points' circle cut off by $MU$. $\therefore$ the angle $QCA = MXU$. Since $M$ bisects $PQ$, $\therefore$ $MX = MD$. $\therefore$ the angle $MXU = QDM$. $\therefore$ $QCA = QDM$. $\therefore$ a circle will go round $QCDE$. $\therefore$ $QEC$ is a right angle. $\therefore$ $E$ is the foot of the perpendicular from $Q$ on $AC$, and is $\therefore$ a point on the pedal line. $\therefore$ the pedal line bisects $PQ$.

For other solutions, see *Casey*, p. 36, and *Catalan*, p. 37.

6. The axes bisect the angle between the asymptotes. $\therefore$ their equation is $\dfrac{x^2 - y^2}{2} = \dfrac{2xy}{\lambda}$. $\therefore$ eliminating $\lambda$, we have for the locus of the vertices,

$$(x^2 - y^2 - a^2)(x^2 - y^2) = -4x^2y^2,$$

$$i.e.\ (x^2 - y^2)^2 + 4x^2y^2 = a^2(x^2 - y^2),$$

or

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

## Paper XXII.

2. If $\alpha + \beta\sqrt{-1}$ be one impossible root, the other will be $\alpha - \beta\sqrt{-1}$. Let $\gamma$ be the third root.

Then $x^3 + qx + r = (x - \alpha + \beta\sqrt{-1})(x - \alpha - \beta\sqrt{-1})(x - \gamma)$

$$= (x^2 - 2\alpha x + \alpha^2 + \beta^2)(x - \gamma)$$

$$= x^3 - (2\alpha + \gamma)x^2 + (\alpha^2 + \beta^2 + 2\alpha\gamma)x - (\alpha^2 + \beta^2)\gamma,$$

$\therefore$ $\gamma = -2\alpha$, $q = \alpha^2 + \beta^2 + 2\alpha\gamma = \alpha^2 + \beta^2 - 4\alpha^2 = \beta^2 - 3\alpha^2$.

4. The angular points of the new triangle are evidently the centres of the escribed circles of the original triangle. $\therefore$ by Todh. *Trig.* Cap. xvi. Ex. 34, $\Delta' = \frac{abc}{2r} = \frac{Pabc}{4\Delta}$. (1)

Again, if $O_1O_2O_3$ be the angular points of the new triangle,

$$r_1 = BO_1 \cos\frac{B}{2},\ r_3 = BO_3 \cos\frac{B}{2},$$

$$\therefore\ O_1O_3 \cos\frac{B}{2} = r_1 + r_3 = a\,\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} + c\,.\,\frac{\cos\frac{B}{2}\cos\frac{A}{2}}{\cos\frac{C}{2}}$$

$$\therefore\ O_1O_3 = a\,\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}} + c\,\frac{\cos\frac{A}{2}}{\cos\frac{C}{2}} = a\left(\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}} + \frac{\sin\frac{C}{2}}{\sin\frac{A}{2}}\right) = \frac{2a}{\sin A}\cos\frac{B}{2}$$

$$= 4R\cos\frac{B}{2} = \frac{abc}{\Delta}\cos\frac{B}{2} = 4\frac{\Delta'}{P}\cos\frac{B}{2},\ \text{by (1)}$$

$$\therefore\ PP' = 4\Delta'\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right).$$

6. Let the bisectors of the angles $F$ and $G$ meet in $O$. Produce $FO$ to meet $AD$ in $K$. Then the angle $FOG = OGD + OKD = OGD + A + AFO = \frac{1}{2}(\pi - A - B) + \frac{1}{2}(\pi - A - D) + A = \pi - \frac{B+D}{2} = \frac{\pi}{2}$.

7. Let $R$ be the middle point of any chord through the fixed point $O$. Take the asymptotes as axes, and let the chord meet the axis of $x$ in $Q'$, and the axis of $y$ in $P'$. Then since $PP' = QQ'$, $R$ is the middle point of $P'Q'$. Let $(h, k)$ be the coordinates of $O$.

The equation to the chord is $\frac{x}{CQ'} + \frac{y}{CP'} = 1$; also $\frac{h}{CQ'} + \frac{k}{CP'} = 1$,

$$\therefore\ \frac{x-h}{CQ'} + \frac{y-k}{CP'} = 0.$$

Now, if $\xi, \eta$ be the coordinates of $R$, $\xi = \frac{CQ'}{2}$, $\eta = \frac{CP'}{2}$,

$\therefore$ the locus of $R$ is $\frac{\xi - h}{2\xi} + \frac{\eta - k}{2\eta} = 0$.

$$\therefore \xi\eta - \xi \cdot \frac{k}{2} - \eta \cdot \frac{h}{2} = 0, \quad \therefore \left(\xi - \frac{h}{2}\right)\left(\eta - \frac{k}{2}\right) = \frac{hk}{4},$$

$\therefore$ transferring to $\left(\frac{h}{2}, \frac{k}{2}\right)$, we have

$$xy = \frac{hk}{4}, \text{ which represents a similar hyperbola.}$$

This question may also be solved geometrically as follows:

Using the same letters as in the analytical proof. bisect $CO$ in $C'$, and through $C'$ draw $EC'F$ parallel to $P'Q'$, meeting $CP'$ in $E$ and $CQ'$ in $F$. Join $RE$, $RF$. Through $C'$ draw lines parallel to $CP'$ and $CQ'$. Let the first meet $ER$ in $Y$ and $CQ'$ in $K$. Let the second meet $RF$ in $X$, and $CP'$ in $H$.

Then $R$ is the middle point of $PQ$, and $\therefore$ also of $P'Q'$. Also $E$ and $F$ are the middle points of $CP'$, $CQ'$. $\therefore$ $ER$, $FR$ are parallel to the asymptotes. $\therefore$ in the parallelogram $CERF$ the complement $YC'RX$ = complement $HK$ = const.

$\therefore$ $RX \,.\, RY$ = const.

$\therefore$ locus of $R$ is a similar hyperbola, centre $C'$.

*This may also be considered as a particular case of the following general proposition.

In any conic the locus of the middle point of chords which pass through a fixed point $O$ is a similar conic.

Describe a similar and similarly situated conic passing through $O$. Let $POO'P'$ be any chord intersecting this conic in $O$, $O'$ and the original conic in $P$, $P'$. Then since the same diameter bisects both $PP'$ and $OO'$, $\therefore$ $OP = O'P'$; and if $R$ be the middle point of $OO'$ or $PP'$, $OR = \frac{1}{2}OO'$. $\therefore$ since $O$ is a fixed point, the locus of $R$ is a conic similar and similarly situated to either of the other two conics.

## Paper XXIII.

1. Put each expression $= R$.

$$\therefore x^3 - myz = Rx^2 \quad (1)$$

$$y^3 - mzx = Ry^2 \quad (2)$$

$$z^3 - mxy = Rz^2 \quad (3)$$

From (1) and (2), $x^3 - y^3 + mz(x - y) = R(x^2 - y^2)$,

$$\therefore x^2 + xy + y^2 + mz = R(x + y).$$

Similarly $y^2 + yz + z^2 + mx = R(y + z)$,

$\therefore$ subtracting $x^2 - z^2 + y(x - z) + m(z - x) = R(x - z)$,

$$\therefore R = x + y + z - m.$$

2. Let $x$ denote the original number of men, $y$ the required number of days.

Then $xy = 10x + \frac{4}{5} \cdot \frac{15}{16}x(y + 20)$;

$$\therefore y = 10 + \tfrac{3}{4}(y + 20); \quad \therefore y = 100.$$

Also $\dfrac{\frac{3}{2}}{x + 150} = \dfrac{\frac{5}{4}}{x}$; $\therefore 6x = 5(x + 150)$; $\therefore x = 750$.

## Paper XXV.

6. Let $h$, $x$ be the height of the cylinder and cone.

Then the vol. of cylinder : vol. of cone :: $h : \frac{x}{3}$,

$\therefore$ weight of cylinder : weight of cone :: $h : \frac{x}{3}$.

Also, the heights above the base of the C. of G. of the cylinder and cone respectively are $\frac{h}{2}$, $\frac{x}{4}$. Now the cylinder with the lead cone inside it is equivalent to the wooden cylinder + a cone of 8 times the weight of the wooden cone. $\therefore$ since the C. of G. of this coincides with the vertex of the cone, its height above the base is $x$.

$$\therefore x = \frac{h \cdot \frac{h}{2} + 8 \cdot \frac{x}{4} \cdot \frac{x}{3}}{h + \frac{8x}{3}} = \frac{3h^2 + 4x^2}{6h + 16x},$$

$$\therefore 4x^2 + 2xh - h^2 = 0.$$

$$\therefore x = h \cdot \frac{-1 \pm \sqrt{5}}{4} = h \sin 18°, \text{ since } x \text{ is positive.}$$

## Paper XXX.

1. (1) $\dfrac{(1-x)(1-y)}{1-x-y} = \dfrac{1-x-y+xy}{1-x-y} = 1 + xy\{1-(x+y)\}^{-1}$

$$= 1 + xy\{1 + (x+y) + (x+y)^2 + \ldots . \\ + (x+y)^{m+n} + \ldots .\},$$

$\therefore$ coefficient of $x^{m+1}\,y^{n+1}$ = coefficient of $x^m\,y^n$ in $(x+y)^{m+n}$

$$= \frac{(m+n)\,!}{m\,!\,n\,!}.$$

(2) Let $\{(1-z)(1-cz)(1-c^2z)(1-c^3z)\}^{-1} = 1 + A_1z + A_2z^2 + \ldots$

Change $z$ into $cz$,

$\therefore \{(1-cz)(1-c^2z)(1-c^3z)(1-c^4z)\}^{-1} = 1 + A_1cz + A_2c^2z^2 + \ldots$

and the expression on the left $= \dfrac{1-z}{1-c^4z}\ (1 + A_1z + A_2z^2 + \ldots)$,

$\therefore\ (1-z)(1 + A_1z + A_2z^2 + \ldots) = (1-c^4z)(1 + A_1cz + A_2c^2z^2 + \ldots)$,

$\therefore$ equating coefficients of $z^{n-1}$ we have

$$A_{n-1} - A_{n-2} = c^{n-1}A_{n-1} - c^{n+2}A_{n-2},$$

$$\therefore A_{n-1} = \frac{1-c^{n+2}}{1-c^{n-1}}A_{n-2};\ \text{and } A_0 = 1,$$

$$\therefore A_1 = \frac{1-c^4}{1-c};\ A_2 = \frac{1-c^5}{1-c^2}\cdot\frac{1-c^4}{1-c};\ A_3 = \frac{1-c^6}{1-c^3}\cdot\frac{1-c^5}{1-c^2}\cdot\frac{1-c^4}{1-c},\ \&c.$$

$$\therefore \text{evidently } A_{n-1} = \frac{(1-c^{n+2})(1-c^{n+1})(1-c^n)}{(1-c^3)(1-c^2)(1-c)}.$$

5. Let $p_1p_2p_3$ be the feet of the perpendiculars from $P$ on $BC$, $CA$, $AB$; and $q_1q_2q_3$ corresponding points for $Q$. Join $BQ$.

Then $q_1SP_1 = \pi - Sp_1q_1 - Sq_1p_1$,

$$Sp_1q_1 = \frac{\pi}{2} + Pp_1p_3 = \frac{\pi}{2} + PBA, \text{ since } PBp_1p_3 \text{ are on a circle.}$$

And $Sq_1p_1 = \frac{\pi}{2} - Qq_1q_3 = \frac{\pi}{2} - QBA$, since $Qq_1Bq_3$ are on a circle.

$$\therefore q_1Sp_1 = \pi - \left(\frac{\pi}{2} + PBA\right) - \left(\frac{\pi}{2} - QBA\right)$$
$$= QBA - PBA = PBQ = PDQ - BQC = PDQ - BAC.$$

## Paper XXXI.

1. The number of shillings depending on the drawing

$$= 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}.$$

The total number of tickets

$$= 1 + 2 + \ldots + (n^2+1) = \tfrac{1}{2}(n^2+1)(n^2+2),$$

$\therefore$ the value of the expectation

$$= \frac{n^2(n+1)^2}{4} \div \tfrac{1}{2}(n^2+1)(n^2+2) = \frac{n^2(n+1)^2}{2(n^2+1)(n^2+2)}.$$

2. (1) $(1 + x + x^2)^n = P_0 + P_1x + P_2x^2 + \ldots$

Change the sign of $x$.

$\therefore (1 - x + x^2)^n = P_0x^{2n} - P_1x^{2n-1} + P_2x^{2n-2} \ldots$ since $P_0 = P_{2n}$, &c.

$$\therefore (P_0 + P_1x + P_2x^2 + \ldots)(P_0x^{2n} - P_1x^{2n-1} + P_2x^{2n-2} \ldots)$$
$$= (1 + x + x^2)^n (1 - x + x^2)^n = (1 + x^2 + x^4)^n$$
$$= P_0 + P_1x^2 + P_2x^4 + \ldots$$

$\therefore$ equating coefficients of $x^{2n}$, we have

$$P_n = P_0^2 - P_1^2 + P_2^2 \ldots$$

(2) $P_0 + P_1x + P_2x^2 \ldots = (1 + x + x^2)^n$

$$= \left(\frac{1 - x^3}{1 - x}\right)^n = (1 - x^3)^n (1 - x)^{-n}$$
$$= \{1 - nx^3 + \ldots\}\{1 + nx + \ldots\}.$$

$\therefore$ equating coefficients of $x^n$,

$$P_n = \frac{n(n+1)\ldots(n+n-1)}{n\,!} - n \cdot \frac{n(n+1)\ldots(n+n-4)}{(n-3)\,!} + \ldots$$
$$= \frac{1}{(n-1)\,!}\left\{\frac{(2n-1)\,!}{n\,!} - n \cdot \frac{(2n-4)\,!}{(n-3)\,!} + \frac{n(n-1)}{2\,!}\frac{(2n-7)\,!}{(n-6)\,!} \ldots\right\}.$$

5. Let $ABC$ be the plane of the paper. Let $G$ be the foot of the perpendicular from $D$ on this plane. Join $GA$, $GB$, $GC$. Produce $AC$ to meet $BG$ in $F$, and produce $GC$ to meet $AB$ in $E$.

Then the plane $EDG$ is perpendicular to the plane $ABC$, and also to the line $AB$, and $BDG$ is perpendicular to $ABC$, and also to $AF$.

$\therefore$ in the triangle $ABG$, $AF$ is perpendicular to $BG$, and $GE$ to $AB$.

$\therefore$ $BC$ is perpendicular to $AG$. $\therefore$ $AD$ is perpendicular to $BC$, since the plane $ADG$ is perpendicular to the plane $ABC$.

6. Let $LM$ touch the parabola in $R$. Join $SR$.

Then the angle $SLP = SRL = \pi - SRM = \pi - SMQ = SMT$, and $SPL = STM$, $\therefore$ the triangles $SPL$ and $STM$ are similar.

Similarly $SLT$ and $SMQ$ are similar triangles.

$\therefore$ $TL : QM :: LS : MS$ from similar triangles $SLT$, $SMQ$,

$:: LP : MT$ ,, ,, $SPL$, $STM$.

## Paper XXXII.

1. Let $p_1$, $p_2$, $p_3$ be the respective probabilities.

Then $$p_1p_2 = \frac{1}{3},\ p_2p_3 = \frac{2}{5},\ p_3p_1 = \frac{1}{10},$$

$$\therefore\ (p_1p_2p_3)^2 = \frac{1}{3 \cdot 5^2},\ \therefore\ p_1p_2p_3 = \frac{1}{5\sqrt{3}},$$

$\therefore$ the odds against all three happening are $5\sqrt{3} - 1$ to 1.

2. If the points $F$, $A$, $E$ are collinear, then

$$\Delta OEF = \Delta OAE + \Delta OAF.$$

Now $\Delta OEF = \frac{1}{2}OE \cdot OF \sin EOF$

$= \frac{1}{2}(m - 1)(n - 1)\ OB \cdot OC \sin BOC$

$= (m - 1)(n - 1)\ \Delta OBC.$

$\Delta OAE = \frac{1}{2}OE \cdot OA \sin AOE = (m - 1)\ \Delta AOB.$

$\Delta OAF = \frac{1}{2}OA \cdot OF \sin AOF = (n - 1)\ \Delta AOC.$

$\therefore$ $(m - 1)(n - 1)\ \Delta BOC = (m - 1)\ \Delta AOB + (n - 1)\ \Delta AOC.$

But $\Delta BOC = \Delta AOB = \Delta AOC$. since $O$ is the C. of G.

$$\therefore\ (m - 1)\ (n - 1) = m - 1 + n - 1,$$

$$\therefore\ (m - 1)\ (n - 1 - 1) = n - 1 = n - 2 + 1,$$

$\therefore (m-2)(n-2) = 1$
Similarly $(n-2)(l-2) = 1$
$(l-2)(m-2) = 1$
} To solve these equations, multiply them together, extract the square root of the product, and divide by each equation.

$\therefore l - 2 = m - 2 = n - 2 = 1. \quad \therefore l = m = n = 3.$

5. Join $CB$. Then $EF$ and $GH$ are each of them parallel to $BC$, and $\therefore$ also to one another, and $\therefore$ a plane can be drawn through them.

$\therefore FG$ and $EH$, being in this plane, intersect.

## Paper XXXIII.

1. Let $x$, $y$, $z$ be the rates per hour of $A$, $B$, and the river respectively.

$A$ takes $\frac{12}{x-z}$ hours to go from $Q$ to $P$, and $\frac{12}{x+z}$ hours from $P$ to $Q$.

$$\therefore \frac{1}{12} + \frac{1}{x-z} = \frac{1}{y-z} \cdots (1)$$
$$\frac{1}{12} + \frac{12}{x-z} + \frac{2}{x+z} = \frac{10}{y-z} \cdots (2)$$
$$\frac{7.2}{x-z} + \frac{12}{x+z} = \frac{6}{y-z} - \frac{1}{2} \cdots (3)$$

These are three simultaneous equations of the first degree, the unknowns being $\frac{1}{x+z}$, $\frac{1}{x-z}$, $\frac{1}{y-z}$. From them we find

$$x + z = 8, x - z = 4, y - z = 3. \quad \therefore x = 6, y = 5, z = 2.$$

$\therefore A$ overtakes $B$ at $(\frac{1}{12} + \frac{1}{4})$ hours after 12 o'clock, *i.e.* at 12.20, and $A$ meets $B$ again at $\frac{1}{3}^0$ hours after 12, *i.e.* at 3.20.
The rate of the stream is 2 miles per hour.

5. The points $G$ and $H$ are on the semicircle described on $AB$ as diameter. Bisect $AB$ at $C$. Then $C$ is the centre of the semicircle.
$\therefore$ the angle $GCH = 2GAH = 2 \times$ comp. of $APG$ = constant, since the angle $APB$ is constant.
$\therefore GH$ is a chord of the semicircle of constant length, and $\therefore$ always touches a circle, centre $C$.

## Paper XXXIV.

1. The necessary condition that we should meet is, that when I start, the policeman must be either in the street or within such a distance of the end of it that he will reach it before I do, *i.e.* he must be within $\frac{1}{m}$th of the length of the street from it. $\therefore$ he must be in a particular $\frac{1}{n}\left(1 + \frac{1}{m}\right)$th of his beat, and the chance of this is $\frac{1+m}{mn}$.

If I walk in the same direction as the policeman, when I start, the policeman must be within $\left(1 - \frac{1}{m}\right)$th of the length of the street from the starting point. The chance of this is $\frac{m-1}{mn}$. $\therefore$ the chance of our meeting (the condition in italics being removed) is

$$\tfrac{1}{2}\left(\frac{m+1}{mn} + \frac{m-1}{mn}\right) = \frac{2m}{2mn} = \frac{1}{n}.$$

2.
$$\left.\begin{aligned} a\frac{x}{z} - b + c\frac{z}{x} &= 0 \\ a\frac{x}{y} + b\frac{y}{x} - c &= 0 \end{aligned}\right\}$$

$$\therefore \frac{a}{1 - \frac{yz}{x^2}} = \frac{b}{\frac{z}{y} + \frac{x}{z}} = \frac{c}{\frac{y}{z} + \frac{x}{y}}.$$

$$\therefore \frac{y}{z}\left(\frac{z}{y} + \frac{x}{z}\right) + \frac{z}{y}\left(\frac{y}{z} + \frac{x}{y}\right) = 1 - \frac{yz}{x^2}.$$

$$\therefore x^{-3} + y^{-3} + z^{-3} + x^{-1}y^{-1}z^{-1} = 0 \quad (A).$$

Again, in the first equation for $x$ writing $\frac{1}{ax}$, &c., we obtain

$$b\frac{cz}{by} + c\frac{by}{cz} = a, \quad \therefore b\frac{y}{z} + c\frac{z}{y} = a.$$

$\therefore$ the equation is unaltered.

$\therefore$ from $(A)$, $a^3x^3 + b^3y^3 + c^3z^3 + abc\,xyz = 0$.

Now multiply the three equations together.

$$\therefore\ \therefore\ abc = \left(b\frac{y}{z} + c\frac{z}{y}\right)\left(ca\frac{z}{y} + cb\frac{yz}{x^2} + a^2\frac{x^2}{yz} + ab\frac{y}{z}\right)$$

$$= 2abc + a\left(b^2\frac{y^2}{z^2} + c^2\frac{z^2}{y^2}\right) + b\left(c^2\frac{z^2}{x^2} + a^2\frac{x^2}{z^2}\right) + c\left(a^2\frac{x^2}{y^2} + b^2\frac{y^2}{x^2}\right)$$

$$= 2abc + a(a^2 - 2bc) + b(b^2 - 2ca) + c(c^2 - 2ab)$$

$$= a^3 + b^3 + c^3 - 4abc.$$

4. Let $P$ be the orthocentre, $I$ the centre of the inscribed circle.

Then $IA = \dfrac{r}{\sin\frac{A}{2}}$, $AP = 2R\cos A$,

the angle $IAP = \dfrac{\pi}{2} - B - \dfrac{A}{2} = \dfrac{C - B}{2}$,

$r = \dfrac{S}{s}$, $R = \dfrac{abc}{4S}$, $\therefore\ r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.

$$\therefore\ IP^2 = IA^2 + AP^2 - 2IA\,.\,AP\cos IAP$$

$$= \frac{r^2}{\sin^2\frac{A}{2}} + 4R^2\cos^2 A - 4\frac{Rr\cos A}{\sin\frac{A}{2}}\cos\frac{B - C}{2}$$

$$= 4R^2\left\{4\sin^2\frac{B}{2}\sin^2\frac{C}{2} + \cos^2 A - 4\sin\frac{B}{2}\sin\frac{C}{2}\cos A\cos\frac{B - C}{2}\right\}$$

$$= 4R^2\left\{(1 - \cos B)(1 - \cos C) + \cos^2 A - \cos A\sin B\sin C - 4\cos A\sin^2\frac{B}{2}\sin^2\frac{C}{2}\right\}$$

$$= 4R^2\left\{(1 - \cos B)(1 - \cos C) - \cos A\cos B\cos C - \cos A(1 - \cos B)(1 - \cos C)\right\}$$

$$= 4R^2\left\{(1 - \cos B)(1 - \cos C)(1 - \cos A) - \cos A\cos B\cos C\right\}.$$

$$\therefore\ IP = 2R\left\{\text{vers}\,A\,\text{vers}\,B\,\text{vers}\,C - \cos A\cos B\cos C\right\}^{\frac{1}{2}}.$$

## Paper XXXV.

2. (1). $\sqrt[4]{(x^2-1)}\left\{\sqrt{x^2+x}-\sqrt{x^2-x}\right\} = \dfrac{x-\sqrt{x^2-1}}{\sqrt{x+\sqrt{x^2-1}}}$

$$= \left\{x-\sqrt{x^2-1}\right\}^{\frac{3}{2}},$$

$$\therefore \sqrt{x^2-1}\left\{2x^2-2x\sqrt{x^2-1}\right\} = (x-\sqrt{x^2-1})^3,$$

$$\therefore 2x^2\sqrt{x^2-1}-2x(x^2-1) = x^3-3x^2\sqrt{x^2-1}+3x(x^2-1)$$
$$-(x^2-1)\sqrt{x^2-1},$$

$$\therefore 6x^3-5x = (6x^2-1)\sqrt{x^2-1},$$

$$\therefore 36x^6-60x^3+25x^2 = (x^2-1)(36x^4-12x^2+1),$$

$$\therefore 12x^4-12x^2-1 = 0,$$

$$\therefore x^2 = \frac{\sqrt{3}\pm 2}{2\sqrt{3}}, \qquad \therefore x = \pm\sqrt{\frac{\sqrt{3}\pm 2}{2\sqrt{3}}}.$$

(2). From (1) and (3), $1+a^3 = x\left\{(y-z)^2+\frac{3}{4}\right\}$

From (2), $= x\left\{\left(\frac{3}{2}-x\right)^2+\frac{3}{4}\right\}$,

$$\therefore x^3-3x^2+3x-1 = a^3, \quad \therefore x-1 = a, \quad \therefore x = a+1,$$

$$\therefore (y+z)^2 = \frac{1+a^3}{1+a} = 1-a+a^2, \quad \therefore y+z = \pm\sqrt{1-a+a^2}$$

$(y-z)^2 = 1-a+a^2-\frac{3}{4} = (a-\frac{1}{2})^2$, $\therefore y-z = \pm(a-\frac{1}{2})$,

$\therefore y$ and $z$ are known.

(3). $a(y-z)+b(z-x)+c(x-y) = 0$ }
and $(y-z)+(z-x)+(x-y) = 0$ }

$$\therefore \frac{y-z}{b-c} = \frac{z-x}{c-a} = \frac{x-y}{a-b} = R,$$

$$\therefore d^3 = (b-c)(c-a)(a-b)R^3. \quad \therefore R \text{ is known.}$$

$$x-y = (a-b)R,\ 2y-2z = (2b-2c)R,$$

$$\therefore x+y-2z = (a+b-2c)R,$$

and $x+y+z = e$, $\therefore z = \frac{1}{3}\left\{e-(a+b-2c)R\right\}$.

Similarly $x = \frac{1}{3}\left\{e-(b+c-2a)R\right\}$, $y = \frac{1}{3}\left\{e-(c+a-2b)R\right\}$.

3. Let $C = 1 + n\frac{a}{c}\cos B + \frac{n(n+1)}{2\,!}\frac{a^2}{c^2}\cos 2B + \ldots$

$$S = \quad n\frac{a}{c}\sin B + \frac{n(n+1)}{2\,!}\frac{a^2}{c^2}\sin 2B + \ldots$$

$$\therefore\ C + Si = 1 + n\frac{a}{c}e^{Bi} + \frac{n(n+1)}{2\,!}\frac{a}{c^2}e^{2Bi} + \ldots$$

$$= \left(1 - n\frac{a}{c}e^{Bi}\right)^{-n}$$

$$= c^n\,(c - a\cos B - i\,.\,a\sin B)^{-n}$$

$$= c^n\,(b\cos A - i\,.\,b\sin A)^{-n}$$

$$= \frac{c^n}{b^n}\,(\cos nA + i\sin nA).$$

$$\therefore\ \frac{\cos nA}{b^n} = \frac{1}{c^n}\,C = \frac{1}{c^n}\left\{1 + n\frac{a}{c}\cos B + \frac{n(n+1)}{2\,!}\frac{a^2}{c^2}\cos 2B + \ldots\right\}.$$

## Paper XLIII.

6. Let $S$, $S'$ be the foci of the ellipse, $H$, $H'$ those of the hyperbola. Let $B$ be the extremity of the conjugate axis of the ellipse, $b$ that of the hyperbola. Let $P$ be a point of intersection, and let the tangents to the ellipse and hyperbola at $P$ meet the conjugate axis in $T$, $t$, and draw $Pn$ perpendicular to $CB$.

Then $CH^2 - CB^2 = CS^2$, $\therefore\ CB^2 = CH^2 - CS^2$.

And $CS^2 + Cb^2 = CH^2$, $\therefore\ Cb^2 = CH^2 - CS^2$, $\therefore\ CB^2 = Cb^2$.

$\therefore\ Cn\,.\,CT = CB^2 = Cb^2 = Cn\,.\,Ct$, $\therefore\ CT = Ct$.

## Paper XLIV.

1. $(x-\theta_1)^2 = (y-\theta_1)(z-\theta_1)$; $(y-\theta_2)^2 = (z-\theta_2)(x-\theta_2)$;

$$(z-\theta_3)^2 = (x-\theta_3)(y-\theta_3);$$

$$\therefore\ x^2 - 2x\theta_1 + \theta_1^2 = yz - \theta_1(y+z) + \theta_1^2,$$

$$\therefore\ \theta_1 = \frac{x^2 - yz}{2x - y - z},$$

$$\therefore\ x - \theta_1 = \frac{2x^2 - xy - xz - x^2 + yz}{2x - y - z} = \frac{(x-y)(x-z)}{2x - y - z}.$$

$\therefore$ Similarly $y - \theta_2 = \dfrac{(y-z)(y-x)}{2y-z-x}$; $z - \theta_3 = \dfrac{(z-x)(z-y)}{2z-x-y}$;

$$\therefore \frac{1}{\theta_1 - x} + \frac{1}{\theta_2 - y} + \frac{1}{\theta_3 - z} = \frac{2x-y-z}{(x-y)(z-x)} + \frac{2y-z-x}{(y-z)(x-y)} + \frac{2z-x-y}{(z-x)(y-z)}.$$

If we reduce the right-hand side to a common denominator, the numerator

$$= 2x(y-z) - (y+z)(y-z) + 2y(z-x) - (z+x)(z-x) + 2z(x-y) - (x+y)(x-y) = 0.$$

4. Take $O$ the centre of the circle. Then $OQ$ is perpendicular to $CD$. $\therefore$ circles will go round $TAQO$ and $TQOB$.

$\therefore$ the angle $AQT = AOT = TOB = TQB$, *i.e.* $TQ$ bisects the angle $AQB$.

Again, since $TAQB$ is a quadrilateral in a circle,

$$TQ \,.\, AB = TB \,.\, AQ + TA \,.\, BQ = TA(AQ + BQ).$$

$$\therefore TQ = \frac{TA}{AB}(AQ + BQ). \quad \therefore TQ \propto AQ + BQ.$$

5. Let $TP$, $TQ$ be two tangents to a parabola, $CD$ a third tangent. Then the circle round $TCD$ always passes through $S$. Let $CD$ change its position, always touching the parabola, until it is indefinitely near to the position $TQ$. Then in its limiting position the circle will pass through two consecutive points on $TP$, and will $\therefore$ touch $TP$ at $T$.

## Paper LI.

3. Let $y = \sqrt{a^2\cos^2\phi + b^2\sin^2\phi} + \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}$,

$$\therefore y^2 = a^2 + b^2 + 2\sqrt{(a^4+b^4)\sin^2\phi\cos^2\phi + a^2b^2(\sin^4\phi + \cos^4\phi)}$$

$$= a^2 + b^2 + 2\sqrt{(a^4+b^4)\cdot\frac{\sin^2 2\phi}{4} + a^2b^2(1 - 2\sin^2\phi\cos^2\phi)}$$

$$= a^2 + b^2 + \sqrt{(a^2-b^2)^2\sin^2 2\phi + 4a^2b^2}$$

$\therefore y^2$ has its max. value when $\sin^2 2\phi = 1$.

$\therefore y^2 = a^2 + b^2 + \sqrt{(a^2-b^2)^2 + 4a^2b^2} = 2(a^2 + b^2), \therefore y = \sqrt{2a^2+2b^2},$
$y^2$ has its min. value when $\sin^2 2\phi = 0,$
$\therefore y^2 = a^2 + b^2 + 2ab, \qquad \therefore y = a + b.$

Next let $y = \dfrac{1}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} + \dfrac{1}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}}$

$$= \frac{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi} + \sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}{\sqrt{\left\{(a^2-b^2)\dfrac{\sin^2 2\phi}{4} + a^2b^2\right\}}},$$

$$\therefore y^2 = \frac{a^2 + b^2 + \sqrt{(a^2-b^2)\sin^2 2\phi + 4a^2b^2}}{(a^2-b^2)\dfrac{\sin^2 2\phi}{4} + a^2b^2}$$

$$= \frac{4(a^2+b^2)}{(a^2-b^2)\sin^2 2\phi + 4a^2b^2} + \frac{4}{\sqrt{(a^2-b^2)\sin^2 2\phi + 4a^2b^2}}.$$

$\therefore y^2$ has its min. value when $\sin^2 2\phi = 1.$

$$\therefore y^2 = \frac{4(a^2+b^2)}{(a^2-b^2)^2 + 4a^2b^2} + \frac{4}{\sqrt{(a^2-b^2)^2 + 4a^2b^2}} = \frac{4}{a^2+b^2} + \frac{4}{a^2+b^2}$$

$$= \frac{8}{a^2+b^2}. \quad \therefore y = \frac{4}{\sqrt{2a^2+2b^2}}$$

$y^2$ has its max. value when $\sin^2 2\phi = 0,$

$$\therefore y^2 = \frac{a^2+b^2}{a^2b^2} + \frac{2}{ab} = \left(\frac{a+b}{ab}\right)^2. \quad \therefore y = \frac{a+b}{ab} = \frac{1}{a} + \frac{1}{b}.$$

5. In a circle if $Pa$ be drawn from a point $P$ on the circumference perpendicular to the side $BC$ of an inscribed triangle, it is parallel to $Oa$, the line joining the centre to the middle point of $BC$. $\therefore$ the required theorem is obtained by projecting orthogonally the proposition 'If perpendiculars be drawn from a point on a circle to the sides of an inscribed triangle, their feet are collinear.'

## Paper LV.

3. Let $a$, $a + b$, $a + 2b$ be the lengths of the sides of the 1st triangle. Then $a + b$ is the length of each side of the equilateral triangle.

The area of the 1st $= \sqrt{\frac{3(a+b)}{2} \cdot \frac{a+3b}{2} \cdot \frac{a+b}{2} \cdot \frac{a-b}{2}}$

$$= \frac{\sqrt{3}}{4}(a+b)\sqrt{(a+3b)(a-b)}.$$

The area of the equilateral triangle $= \frac{\sqrt{3}}{4}(a+b)^2$,

$$\therefore \sqrt{(a+3b)(a-b)} : a+b :: 3 : 5,$$

$$\therefore 25(a^2+2ab-3b^2) = 9(a^2+2ab+b^2),$$

$$\therefore 16a^2+32ab-84b^2 = 0, \quad \therefore b = \tfrac{2}{3}a, \text{ or } -\tfrac{2}{7}a.$$

If $A$ be the angle opposite the side $a+2b$, then taking $b = \frac{2}{3}a$,

$$\cos A = \frac{a^2+(a+b)^2-(a+2b)^2}{2a(a+b)} = \frac{a^2+\frac{25}{9}a^2-\frac{49}{9}a^2}{2\cdot\frac{3}{7}\cdot\frac{5}{7}\cdot a^2} = -\tfrac{1}{2}, \quad \therefore A = 120°.$$

If we take $b = -\frac{2}{7}a$, the greatest side is $a$,

$$\therefore \cos A = \frac{(a+2b)^2+(a+b)^2-a^2}{2(a+2b)(a+b)} = \frac{a^2\cdot\frac{9}{49}+a^2\cdot\frac{25}{49}-a^2}{2\cdot\frac{3}{7}\cdot\frac{5}{7}\cdot a^2} = -\tfrac{1}{2}, \therefore A = 120°.$$

5. Let $PAB$ be the secant required. Join $QA$, $QB$. Then the angles $PAQ$ and $PBC$ are right angles, $\therefore AQ$ is parallel to $BC$. $\therefore \triangle ABC = QBC$. Now the triangle $QBC$ is a max. when $B$ is the middle point of the arc $PBC$. $\therefore PBC$ is an isosceles right-angled triangle, as is also $PAQ$. $\therefore A$ is the middle point of the arc $PAQ$. $\therefore$ the triangle $QAC$ is a max. and $QAC = QAB$.

## Paper LVII.

2. $\frac{p_2}{q_2} = \frac{1-x^3}{(1-x)(1-x^3)+x} = \frac{1-x^3}{1-x^3+x^4} = \frac{x^{-1}-x^{-4}}{x^{-1}-x^{-4}-1} = \frac{\sigma_2}{\sigma_2-1}$

$$\frac{p_3}{q_3} = \frac{(1-x^3)(1-x^5)+x^3}{(1-x)(1-x^3)(1-x^5)+x^3(1-x)+x(1-x^5)}$$

$$= \frac{1-x^5+x^3}{1-x^5+x^3-x^9} = \frac{x^{-1}-x^{-4}+x^{-9}}{x^{-1}-x^{-4}+x^{-9}-1} = \frac{\sigma_3}{\sigma_3-1}.$$

Assume that the law holds for $\frac{p_{n-2}}{q_{n-2}}$ and $\frac{p_{n-1}}{q_{n-1}}$.

Then $p_{n-2} = (-1)^{n-1} . x^{(n-2)^2} \sigma_{n-2}$, $q_{n-2} = (-1)^{n-1} . x^{(n-2)^2} (\sigma_{n-2} - 1)$,

$p_{n-1} = (-1)^n \quad . x^{(n-1)^2} \sigma_{n-1}$, $q_{n-1} = (-1)^n \quad . x^{(n-1)^2} (\sigma_{n-1} - 1)$,

$$\therefore \frac{p_n}{q_n} = \frac{(1 - x^{2n-1})\, x^{(n-1)^2} \sigma_{n-1} - x^{2n-3} . x^{(n-2)^2} \sigma_{n-2}}{(1 - x^{2n+1})\, x^{(n-1)^2} (\sigma_{n-1} - 1) - x^{2n-3} . x^{(n-2)^2} (\sigma_{n-2} - 1)}$$

$$= \frac{x^{n^2-2n+1} (\sigma_{n-1} - \sigma_{n-2}) - x^{n^2} \sigma_{n-1}}{x^{n^2-2n+1} (\sigma_{n-1} - \sigma_{n-2}) - x^{n^2} (\sigma_{n-1} - 1)}. \quad \text{Divide by } x^{n^2}.$$

$$= \frac{\sigma_{n-1} + x^{-n^2}}{\sigma_{n-1} + x^{-n^2} - 1} = \frac{\sigma_n}{\sigma_n - 1}.$$

And the law has been shewn to hold when $n = 2$, $n = 3$, $\therefore$ it holds universally.

7. Let $S$ be the focus, $PT$ the tangent at any point $P$. Draw $ST$ making the angle $STP$ equal to the given angle. Draw $SY$ perpendicular to $PT$. Then the angle $YST$ is constant, $\therefore$ $SY : ST$ in a constant ratio. And since the locus of $Y$ is a circle, $\therefore$ the locus of $T$ is a circle. See Ap. Note 1, §§ 4, 5.

## Paper LIX.

5. $$\frac{BD}{CD} = \frac{BQ . OR}{QO . RC}; \quad \frac{CE}{AE} = \frac{CR . OP}{RO . PA}; \quad \frac{AF}{BF} = \frac{AP . OQ}{PO . QB};$$

$$\therefore \frac{BD . CE . AF}{CD . AE . BF} = 1. \quad \therefore AD, BE, CF \text{ are concurrent.}$$

6. Let $AB$ be the given diameter, $S$ the fixed point in it, $P$ any point on the semicircle. Then the line of the crease bisects $SP$ at right angles in $Y$, and $SY : SP :: 1 : 2$. Now the locus of $P$ is a circle, $\therefore$ the locus of $Y$ is a circle. $\therefore$ the line of the crease envelopes a conic which has $S$ for a focus, and the locus of $Y$ for auxiliary circle. The conic is an ellipse or hyperbola according as $P$ is in $AB$ or $AB$ produced. If $P$ coincides with $A$ or $B$, the envelope is the point at the other extremity of the diameter.

7. (1). Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation of the ellipse, and let $\theta, \phi$ be the eccentric angles of the extremities of a chord of length $2l$.

Then $$4l^2 = a^2(\cos\theta - \cos\phi)^2 + b^2(\sin\theta - \sin\phi)^2,$$

$$\therefore l^2 = \left(a^2 \sin^2 \frac{\theta + \phi}{2} + b^2 \cos^2 \frac{\theta + \phi}{2}\right) \sin^2 \frac{\theta - \phi}{2}.$$

If $(x, y)$ be the coordinates of the middle point of the chord,

$$x = \tfrac{1}{2}a(\cos\theta + \cos\phi) = a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

$$y = \tfrac{1}{2}b(\sin\theta + \sin\phi) = b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

$$\therefore b^2x^2 + a^2y^2 = a^2b^2\cos^2\frac{\theta-\phi}{2},$$

and $b^4x^2 + a^4y^2 = a^2b^2\left(a^2\sin^2\dfrac{\theta+\phi}{2} + b^2\cos^2\dfrac{\theta+\phi}{2}\right)\cos^2\dfrac{\theta-\phi}{2}$

$$= a^2b^2l^2\cot^2\frac{\theta-\phi}{2},$$

$$\therefore \frac{1}{b^2x^2 + a^2y^2} - \frac{l^2}{b^4x^2 + a^4y^2} = \frac{1}{a^2b^2\cos^2\dfrac{\theta-\phi}{2}}\left(1 - \sin^2\frac{\theta-\phi}{2}\right)$$

$$= \frac{1}{a^2b^2}.$$

(2). Let $y^2 = px$ be the equation to the parabola.

Since the coordinates of any point can be put in the form $(pm^2, pm)$, let the coordinates of the extremities of a chord of length $2l$ be $(pm_1^2, pm_1)$ and $(pm_2^2, pm_2)$.

Then $4l^2 = p^2(m_1^2 - m_2^2)^2 + p^2(m_1 - m_2)^2$

$$= p^2(m_1 - m_2)^2\{(m_1 + m_2)^2 + 1\}.$$

If $(x, y)$ be the coordinates of the middle point of the chord,

$$2x = p(m_1^2 + m_2^2), \quad 2y = p(m_1 + m_2),$$

$$\therefore 4(y^2 - px) = p^2(m_1^2 + 2m_1m_2 + m_2^2 - 2m_1^2 - 2m_2^2)$$

$$= -p^2(m_1 - m_2)^2,$$

and $$4y^2 + p^2 = p^2\{(m_1 + m_2)^2 + 1\},$$

$$\therefore 16(y^2 - px)\left(y^2 + \frac{l^2}{4}\right) = -p^4(m_1 - m_2)^2\{(m_1 + m_2)^2 + 1\}$$

$$= -4p^2l^2.$$

## PAPER LX.

2. The $n^{\text{th}}$ term of the given series $= 4(n-1)+1$.

$\therefore$ the sum of $n$ terms $= 2(n-1)n + n = 2n^2 - n = y^2$ suppose.

$$\therefore 2\left(n^2 - \frac{n}{2} + \tfrac{1}{16}\right) - y^2 = \tfrac{1}{8},$$

$$\therefore 16(n - \tfrac{1}{4})^2 - 8y^2 = 1,$$

or $$x^2 - 8y^2 = 1, \text{ where } x = 4n - 1.$$

$x = 3$, $y = 1$ is obviously one solution,

$$\therefore (x - \sqrt{8}y)(x + \sqrt{8}y) = 1 = (3 - \sqrt{8})^m (3 + \sqrt{8})^m.$$

Let $x - \sqrt{8}y = (3 - \sqrt{8})^m$, $\therefore x + \sqrt{8}y = (3 + \sqrt{8})^m$,

$$\therefore 2x = (3 - \sqrt{8})^m + (3 + \sqrt{8})^m.$$

By giving $m$ different values, we can obtain as many values as we please of $x$, and $\therefore$ also of $n$.

If $m = 1$, $x = 3$, $4n = 4$, $\therefore n = 1$.

$m = 2$, $x = 17$, $4n = 18$, inadmissible.

$m = 3$, $x = 27 + 72$, $4n = 100$, $\therefore n = 25$.

$m = 4$, $x = 81 + 432 + 64$, $4n = 578$, inadmissible.

$m = 5$, $x = 243 + 2160 + 960$, $4n = 3364$, $\therefore n = 841$.

$\therefore$ the first two values of $n$ greater than unity are 25, 841.

4. Draw $TK$ a tangent to the inner circle, and produce $RPQ$ to meet the outer circle in $S$. Join $AS$.

Then $RP : RQ :: RP^2 : RP \cdot RQ :: RP^2 : TK^2 :: RP^2 : AQ^2$, since $TK = AQ$, being tangents from points on the outer circle.

$\therefore AR = AS$. $\therefore$ the angle $RTP = ARQ$, and $RPT = APQ = AQP$.

$$\therefore RT^2 : RA^2 :: RP^2 : AQ^2 :: RP : RQ.$$

## PAPER LXI.

4. Let $P$ be the given point, $RE$ the polar of $P$, $C'PC$ any chord through $P$. Produce $C'C$ to meet $RE$ in $R$. Draw $CD$, $C'D'$, $PE$ perpendiculars on $RE$. Then $EP$ passes through the centre $O$. Let the tangents at $C$, $C'$ meet in $H$ on $RE$. Join $CO$. Draw $PQ$, $PQ'$, $PG$, $CF$ perpendiculars on $HC$, $HC'$, $CO$, $OE$ respectively.

Then $OP \cdot OE = (\text{rad.})^2 = OC \cdot OC$,

$OP \cdot OF = OC \cdot OG$,

$\therefore$ subt. $OP \cdot EF = OC \cdot CG$. $\therefore OP \cdot CD = OC \cdot PQ$. $(A)$

Similarly we can prove $OP \cdot C'D' = OC' \cdot PQ'$. $(B)$

Now since $RCPC'$ is a harmonic range

$$\frac{1}{CR} + \frac{1}{C'R} = \frac{2}{PR},$$

$$\therefore \frac{1}{CD} + \frac{1}{C'D'} = \frac{2}{PE}.$$

$\therefore$ from $(A)$ and $(B)$,

$$OP\left(\frac{1}{PQ} + \frac{1}{PQ'}\right) = OC\left(\frac{1}{CD} + \frac{1}{C'D'}\right) = \frac{2OC}{PE}.$$

$$\therefore \frac{1}{PQ} + \frac{1}{PQ'} = \frac{2OC}{OP \cdot PE} = \text{const.}$$

NOTE.—Reciprocating with respect to $P$ we obtain the theorem: 'The sum of the focal distances of a point on an ellipse is constant.'

## PAPER LXVII.

7. Let $A$, $B$, $C$ be the points on the bar at which the strings from the pulleys are fastened, and let $D$ be the point at which the weight is suspended. Let $CD = x$. Then if $a$, $b$, $c$ be the radii of the pulleys whose strings are fastened at $A$, $B$, $C$,

$$CB = 2c - b,\ BA = 2b - a,$$

$A$ corresponding to the lowest pulley.

Then we have forces $P$, $2P$, $4P$ acting upwards at $A$, $B$, $C$, respectively, and $7P$ downwards at $D$. $\therefore$ taking moments about $C$,

$$4P \cdot CD = 2P \cdot BD + P \cdot AD,$$

$$\therefore 4x = 4c - 2b - 2x + 2c + b - a - x = 6c - b - a - 3x,$$

$$\therefore x = \frac{6c - b - a}{7}.$$

When there is equilibrium, $W = 7P$, $\therefore$ if the power be moving at any time with acceleration $f$, and if $f'$ be the acceleration of the

weight, $f' = \frac{1}{7} . f$. Suppose the power to be $2P$ instead of $P$. Then after any time $t$ the kinetic energy which the system possesses will be equivalent to the work done by the system against gravity in that time.

$$\therefore \tfrac{1}{2}f'^2t^2 . 2P + \tfrac{1}{2}f'^2t^2W = \tfrac{1}{2}ft^2g . 2P - \tfrac{1}{2}f't^2gW.$$

$$\therefore \tfrac{49}{2}f'^2 . 2P + \tfrac{1}{2}f'^2 . 7P = \tfrac{7}{2}fg . 2P - \tfrac{1}{2}f'g . 7P.$$

$$\therefore 2 . 49f'^2 + 7f'^2 = g(14f' - 7f'), \quad \therefore 15f' = g.$$

## PAPER LXX.

2. $$x_1x_2 + y_1y_2 = 1, \quad x_1x_3 + y_1y_3 = 1$$

$$\therefore \frac{x_1}{y_3 - y_2} = \frac{y_1}{x_2 - x_3} = \frac{x_1x_2 + y_1y_2}{x_2y_3 - x_3y_2} = \frac{1}{d_1}.$$

Similarly $$\frac{x_2}{y_1 - y_3} = \frac{y_2}{x_3 - x_1} = \frac{x_2x_3 + y_2y_3}{x_3y_1 - x_1y_3} = \frac{1}{d_2}$$

and $$\frac{x_3}{y_2 - y_1} = \frac{y_3}{x_1 - x_2} = \frac{x_3x_1 + y_3y_1}{x_1y_2 - x_2y_1} = \frac{1}{d_3}.$$

$\therefore$ we have to prove that

$$\frac{x_2 - x_3}{y_1} + \frac{x_3 - x_1}{y_2} + \frac{x_1 - x_2}{y_3} = \frac{(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)}{y_1y_2y_3},$$

$$i.e.\ y_2y_3(x_2 - x_2) + \ldots = (x_2 - x_3)(x_3 - x_1)(x_1 - x_2).$$

The left-hand side

$$= y_2y_3(x_2 - x_3) + y_3y_1(x_3 - x_1) - y_1y_2(x_2 - x_3 + x_3 - x_1)$$

$$= (y_2y_3 - y_1y_2)(x_2 - x_3) + (y_3y_1 - y_1y_2)(x_3 - x_1)$$

$$= y_2(y_3 - y_1)(x_2 - x_3) + y_1(y_3 - y_2)(x_3 - x_1).$$

Now $y_1(y_3 - y_2) = x_1(x_2 - x_3)$; $y_2(y_3 - y_1) = x_2(x_1 - x_3)$,

$\therefore$ the left-hand side

$$= x_2(x_1 - x_3)(x_2 - x_3) + x_1(x_2 - x_3)(x_3 - x_1)$$

$$= (x_2 - x_3)(x_3 - x_1)(x_1 - x_2).$$

This may also be proved as follows.

Let $D = d_1 + d_2 + d_3$,

$$\therefore D = \begin{vmatrix} x_1, y_1, 1 \\ x_2, y_2, 1 \\ x_3, y_3, 1 \end{vmatrix} = \begin{vmatrix} x_1, y_1, 1 - x_1^2 - y_1^2 \\ x_2, y_2, 1 - x_2x_1 - y_2y_1 \\ x_3, y_3, 1 - x_3x_1 - y_3y_1 \end{vmatrix} = \begin{vmatrix} x_1, y_1, 1 - x_1^2 - y_1^2 \\ x_2, y_2, 0 \\ x_3, y_3, 0 \end{vmatrix}$$

$$= (1 - x_1^2 - y_1^2)d_1.$$

$$\text{Now } d_2d_3 = \begin{vmatrix} x_3, y_3 \\ x_1, y_1 \end{vmatrix} \times \begin{vmatrix} x_1, y_1 \\ x_2, y_2 \end{vmatrix} = \begin{vmatrix} x_3x_1 + y_3y_1, & x_2x_3 + y_2y_3 \\ x_1^2 + y_1^2, & x_1x_2 + y_1y_2 \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 1 \\ y_1^2 + y_1^2, & 1 \end{vmatrix} = 1 - x_1^2 - y_1^2$$

$$\therefore d_1 + d_2 + d_3 = d_1d_2d_3.$$

## Paper LXXV.

6. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation of the conic round $ABC$. Let $(h, k)$ be the coordinates of $P$, $(a \cos \alpha, b \sin \alpha)$ the coordinates of $A$.

The equation of $AP$ is $y - b \sin \alpha = \frac{k - b \sin \alpha}{h - a \cos \alpha}(x - a \cos \alpha)$,

$\therefore$ the equation of diameter conjugate to $AP$ is

$$y = -\frac{b^2}{a^2} \cdot \frac{h - a \cos \alpha}{k - b \sin \alpha} \cdot x,$$

$$i.e.\ b^2(h - a \cos \alpha)x + a^2(k - b \sin \alpha)y = 0.$$

The equation of the tangent at $A$ is $\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha = 1$,

$\therefore$ the line whose equation is

$$b^2(h - a\cos\alpha)x + a^2(k - b\sin\alpha)y + ab(bx\cos\alpha + ay\sin\alpha - ab) = 0$$

passes through $D$. This reduces to

$$b^2hx + a^2ky - a^2b^2 = 0, \quad \text{or } \frac{xh}{a^2} + \frac{yk}{b^2} = 1,$$

which is the equation of the polar of $P$. Similarly it may be shewn that the polar of $P$ passes through $E$ and $F$.

## PAPER LXXVI.

7. Let $AD$ be the horizontal line in the side of the wall, $E$ the point of projection. Let $AB$, $BC$, $CD$ be the sections of the walls made by a horizontal plane passing through $AD$, and let $F$, $G$, $H$ be the points of impact in $AB$, $BC$, $CD$ respectively.

Let $DE = a$, $EA = \mu a$, $AEF = \theta$.

Then $AF = \mu a \tan\theta$, $\therefore FB = a + \mu a(1 - \tan\theta)$,

and $\tan BFG = e \cot\theta$.

$$BG = BF \tan BFG = ae \cot\theta + \mu ae (\cot\theta - 1),$$

$$\therefore CG = (\mu + 1)a(1 - e\cot\theta) + \mu ae.$$

$$\tan CGH = e \tan BGF = e \cdot \frac{1}{e} \tan\theta = \tan\theta,$$

$$\therefore CH = (\mu + 1)a(\tan\theta - e) + \mu ae \tan\theta,$$

$$\therefore DH = a(\mu + 1)(1 + e - \tan\theta) - \mu ae \tan\theta.$$

$$\tan DHE = e \tan CHG = e \cot\theta,$$

$$\therefore DE = ae(\mu + 1)\{(1 + e)\cot\theta - 1\} - \mu ae^2 = \mu a,$$

$$\therefore e(\mu + 1)\cot\theta = e\mu + 1.$$

## PAPER LXXVII.

5. Let $P_1$, $P_2$ be the greatest and least forces.

Then $P_1 : W :: \sin\alpha + \mu\cos\alpha : 1$,

and $P_2 : W :: \sin\alpha - \mu\cos\alpha : 1$,

and $W : R :: 1 : \cos\alpha$.

$$\therefore (\sin\alpha + \mu\cos\alpha)\cos\alpha = \sin\alpha - \mu\cos\alpha.$$

$$\therefore \mu = \frac{\sin\alpha(1 - \cos\alpha)}{\cos\alpha(1 + \cos\alpha)} = \tan\alpha \tan^2\frac{\alpha}{2}.$$

## PAPER LXXVIII.

7. Let the projectiles come into collision $t$ seconds after the second is discharged. At the moment of collision their horizontal and vertical distances from the point of projection must be respectively equal.

$$\therefore V\cos\alpha\,.\,(n+t) = V'\cos\alpha'\,.\,t,$$

and $$V\sin\alpha\,.\,(n+t) - \tfrac{1}{2}g(n+t)^2 = V'\,.\,\sin\alpha'\,.\,t - \tfrac{1}{2}gt^2,$$

$$\therefore t(V\sin\alpha - V'\sin\alpha' - gn) = n(\tfrac{1}{2}gn - V\sin\alpha),$$

and $$Vn\cos\alpha = t(V'\cos\alpha' - V\cos\alpha),$$

$$\therefore VV'\sin(\alpha-\alpha') = \tfrac{1}{2}gn(V\cos\alpha + V'\cos\alpha').$$

## Paper LXXIX.

3. (1). Since $\tan x = \cot x - 2\cot 2x$, the given expression on the right

$$= \tan\frac{\pi}{2^{n+1}}\left\{\cot\frac{\pi}{2^{n+1}} - 2\cot\frac{\pi}{2^n} + 2\cot\frac{\pi}{2^n} - 2^2\cot\frac{\pi}{2^{n-1}} \right.$$

$$\left. \cdots\cdots\cdots\cdots\cdots\cdots\cdots \right.$$

$$\left. + 2^{n-2}\cot\frac{\pi}{2^3} - 2^{n-1}\cot\frac{\pi}{2^2} + 2^{n-1}\right\}$$

$$= \tan\frac{\pi}{2^{n+1}}\left\{\cot\frac{\pi}{2^{n+1}} - 2^{n-1}\cot\frac{\pi}{4} + 2^{n-1}\right\} = 1.$$

*(2). This question should be stated as follows. Prove that

$$2 = \frac{1}{\cos^2\frac{\theta}{2}} + \frac{1}{2}\cdot\frac{\cos\theta}{\cos^2\frac{\theta}{2}\cos^2\frac{\theta}{2^2}} + \frac{1}{2^2}\frac{\cos\theta\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}\cos^2\frac{\theta}{2^2}\cos^2\frac{\theta}{2^3}} + \cdots$$

Since $2\cos^2\frac{\theta}{2} = 1 + \cos\theta$,

$$\therefore 2 = \frac{1}{\cos^2\frac{\theta}{2}} + \frac{\cos\theta}{\cos^2\frac{\theta}{2}} = \frac{1}{\cos^2\frac{\theta}{2}} + \frac{1}{2}\cdot\frac{\cos\theta}{\cos^2\frac{\theta}{2}}\cdot 2.$$

Similarly, by changing $\theta$ into $\frac{\theta}{2}, \frac{\theta}{2^2}\ldots$

$$2 = \frac{1}{\cos^2\frac{\theta}{2^2}} + \frac{1}{2}\cdot\frac{\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2^2}}\cdot 2.$$

$$2 = \frac{1}{\cos^2\frac{\theta}{2^3}} + \frac{1}{2}\cdot\frac{\cos\frac{\theta}{2^2}}{\cos^2\frac{\theta}{2^3}}\cdot 2.$$

. . . . . .

$\therefore$ substituting, we have

$$2 = \frac{1}{\cos^2\frac{\theta}{2}} + \frac{1}{2}\cdot\frac{\cos\theta}{\cos^2\frac{\theta}{2}}\left[\frac{1}{\cos^2\frac{\theta}{2^2}} + \frac{1}{2}\cdot\frac{\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2^2}}\left\{\frac{1}{\cos^2\frac{\theta}{2^3}} + \ldots\right\}\right]$$

= given series.

(3) $\operatorname{scc}^2 A + \operatorname{cosec}^2 A = 2^2\operatorname{cosec}^2 2A$; $\therefore \sec^2 A = 2^2\operatorname{cosec}^2 2A - \operatorname{cosec}^2 A$.

$2^2(\sec^2 2A + \operatorname{cosec}^2 2A) = 2^4\operatorname{cosec}^2 2^2 A$; $\therefore 2^2\sec^2 2A = 2^4\operatorname{cosec}^2 2^2 A$

$- 2^2\operatorname{cosec}^2 2A$.

. . . . . .

$$2^{2(n-1)}\,(\sec^2 2^{n-1}A + \operatorname{cosec}^2 2^{n-1}A) = 2^{2n}\operatorname{cosec}^2 2^n A;$$

$$\therefore 2^{2n-1}\sec^2 2^{n-1}A = 2^{2n}\operatorname{cosec}^2 2^n A - 2^{2n-2}\operatorname{cosec}^2 2^{n-1}A.$$

$\therefore$ by addition, the given series $= 2^{2n}\operatorname{cosec}^2 2^n A - \operatorname{cosec}^2 A$.

4. Let $ABC$ be the given triangle, $A'B'C'$ the vertices of the isosceles triangles on $BC$, $CA$, $AB$. Let $AA'$, $BB'$, $CC'$ meet $BC$, $CA$, $AB$ in $\alpha$, $\beta$, $\gamma$. Then since the isosceles triangles are similar, $AB : AC' :: AC : AB'$, and the angle $CAB' = BAC'$, $\therefore$ the angle $CAC' = BAB'$. $\therefore$ by Euc. vi. 15, the triangles $ABB'$, $ACC'$ are equal.

Similarly the triangle $BCC' = BAA'$, and $CAA' = CBB'$.

Now $\quad \Delta\, BAA' : \Delta\, CAA' :: Ba : Ca,$

and $\quad \Delta\, CBB' : \Delta\, ABB' :: C\beta : A\beta,$

and $\quad \Delta\, ACC' : \Delta\, BCC' :: A\gamma : B\gamma,$

$$\therefore Ba \,.\, C\beta \,.\, A\gamma = Ca \,.\, B\gamma \,.\, A\beta,$$

$\therefore$ $Aa$, $B\beta$, $C\gamma$ are concurrent.

5. $\quad (x - a)^2 + y^2 = c^2 \quad (1).$

The equation of a chord through the origin is $y = mx \quad (2)$.

From (1) and (2), $\quad (x - a)^2 + m^2x^2 = c^2,$

$$\therefore (1 + m^2)x^2 - 2ax + a^2 - c^2 = 0.$$

If $(x_1y_1)$, $(x_2y_2)$ be the points where (1) intersects (2), the equation of the circle described on the intercepted chord as diameter is

$$\begin{aligned} 0 &= (x - x_1)(x - x_2) + (y - y_1)(y - y_2) \\ &= x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1x_2 + y_1y_2 \\ &= (x^2 + y^2)(1 + m^2) - 2ax - 2may + (a^2 - c^2)(1 + m^2) \\ &= (x^2 + y^2 + a^2 - c^2)m^2 - 2may + x^2 + y^2 - 2ax + a^2 - c^2. \end{aligned}$$

$\therefore$ expressing the condition that the values of $m$ are equal, we obtain as the equation of the envelope

$$(x^2 + y^2 - 2ax + a^2 - c^2)(x^2 + y^2 + a^2 - c^2) = a^2y^2.$$

Let $d$ be the distance of any point on the envelope from the centre of the given circle.

Then $d^2 = (x - a)^2 + y^2 = x^2 + y^2 + a^2 - 2ax$,

$$\therefore (d^2 - c^2)(d^2 + 2ax - c^2) = a^2y^2 = a^2\{d^2 - (x - a)^2\},$$

$$\therefore a^2x^2 - 2ax(a^2 + c^2 - d^2) + d^4 + c^4 + a^4 - d^2(a^2 + 2c^2) = 0,$$

$$\therefore a^2x = a(a^2 + c^2 - d^2) \pm a\sqrt{N},$$

where $N = a^4 + c^4 + d^4 - 2d^2(a^2 + c^2) + 2a^2c^2 + d^2(a^2 + 2c^2) - d^4 - c^4 - a^4$

$$= 2a^2c^2 - a^2d^2.$$

Now if $x$ is real, $N$ must not be negative. $\therefore$ the max. value which $d$ can have is $c\sqrt{2}$.

## Paper LXXX.

7. Let $v$ be the velocity of projection, $\theta$ the angle which the direction of projection makes with the horizontal. Let $t$, $t'$ be the times taken to reach the net and the service line respectively.

Then $vt\cos\theta = 39$; $\therefore t = \dfrac{39}{v\cos\theta}$;

and $vt\sin\theta + \frac{1}{2}gt^2 = 4\frac{3}{4} = \frac{19}{4}$, $\therefore 156\tan\theta + \dfrac{2 \cdot 39^2}{v^2\cos^2\theta} \cdot g = 19$

$$vt'\sin\theta + \tfrac{1}{2}gt'^2 = 8, \quad \therefore 15\tan\theta + \frac{2 \cdot 15^2}{v^2\cos^2\theta} \cdot g = 2,$$

$$\therefore \frac{2 \cdot 3^2}{v^2 \cos^2\theta} \cdot g = \frac{19 - 156 \tan\theta}{169} = \frac{2 - 15 \tan\theta}{25}, \quad \therefore \tan\theta = \frac{137}{1365}.$$

$$\therefore v^2 \cos^2\theta = \frac{2 \cdot 3^2 \cdot 25 \cdot g}{2 - \frac{15 \times 137}{1365}} = \frac{262050}{11} = 23833 \text{ nearly},$$

$$\therefore v \cos\theta = 160 \text{ nearly}.$$

## PAPER LXXXV.

4. Consider the asymptote as a tangent having its point of contact $L$ at an infinite distance.

Then the angle $STP = HTC$. Bes. *Con.* Hyp. Prop. xii.

$= \text{sup. } HTL = THP$. *Id.* p. 12, Prop. xii.

## PAPER LXXXVIII.

7. Let $T$ be the tension of the string. On the 1st pulley we have $2T$ upwards and $P + T$ downwards. $\therefore 2T = P + T$, $\therefore P = T$.

On the 2nd pulley we have $2T$ upwards and $W + P$ downwards.

$$\therefore 2T = W + P, \quad \therefore W = T = P.$$

If $W$ be greater than $P$, the weight acting on the lower pulley is $W + P$, and at the upper pulley $P$; also if $f$ be the acceleration of the lower pulley, $2f$ is that of the upper pulley, and the kinetic energy of the system at the end of time $t$ is

$$= \tfrac{1}{2}(W + P)f^2t^2 + \tfrac{1}{2}P(2f)^2t^2 = \tfrac{1}{2}(W + 5P)f^2t^2. \quad (1)$$

Now the moving force on the system is $W - P$, and its point of application may be supposed to be at the lowest pulley, and at the end of time $t$ this point has moved through a space $\frac{1}{2}ft^2$, $\therefore$ the work done on the system $= \frac{1}{2}ft^2(W - P)g$. (2)

$\therefore$ equating (1) and (2) we have

$$\tfrac{1}{2}(W + 5P)f^2t^2 = \tfrac{1}{2}ft^2(W - P)g. \quad \therefore f = \frac{W - P}{W + 5P} \cdot g.$$

## PAPER XCII.

7. Let $a$ be the angle which a side $c$ makes with a side $b$. Then we have $c \sin a + d \cos a = a$; (1) $\quad c \cos a + d \sin a = b$; (2)

$$\cot a = e \tan a, \quad \therefore \cot a = \sqrt{e}.$$

From (1) and (2), $(bc - ad) \sin a = (ac - bd) \cos a$,

$$\therefore e = \cot^2 a = \left(\frac{bc - ad}{ac - bd}\right)^2.$$

If the ball moves the opposite way round the rectangle $cd$, we must write $\frac{\pi}{2} - a$ for $a$, and we obtain $e = \left(\frac{ac - bd}{bc - ad}\right)^2$.

## Paper XCIII.

5. Let $PG$ be the normal at $P$. Draw $GZQ$ perpendicular to $PG$, and $SZ$ parallel to $PG$, meeting the tangent at $P$ in $Y$. Then $SY = \frac{1}{2}PG = \frac{1}{2}YZ$. $\therefore YS = SZ$. Now the locus of $Y$ is the tangent at the vertex of the given parabola. $\therefore$ the locus of $Z$ is a straight line $ZA'$ perpendicular to the axis, and $SA' = SA$. $\therefore$ since $S$ is a fixed point and $SZQ$ a right angle, the envelope of $QG$ is a parabola equal to the given parabola, having the same focus, but its concavity in the opposite direction.

Produce $PS$ to meet $GZ$ in $Q$. Then from the equal triangles $SYP$, $SZQ$, $SQ = SP = SG$. $\therefore G$ is the point on the envelope corresponding to the point $P$ on the given parabola.

6. From the first 3 equations we see that $(x_1y_1)$, $(x_2y_2)$, $(x_3y_3)$ are the roots of the equation $x^3 + y^3 = a^3$ $(A)$. $\therefore$ regarding this as the equation of a curve, (4) tells us that the three points given by these roots are collinear. Let the equation to the line on which they are be

$$y = mx + b \quad (B).$$

Substitute the value of $y$ from $(B)$ in $(A)$,

$$\therefore x^3 + (b - mx)^3 = a^3.$$

$$\therefore x^3(1 - m^3) + 3m^2bx^2 - 3mb^2x - (a^3 - b^3) = 0.$$

$$\therefore x_1x_2x_3 = \frac{a^3 - b^3}{1 - m^3}.$$

Substitute the value of $x$ from $(B)$ in $(A)$,

$$\therefore m^3y^3 + (b - y)^3 = a^3m^3.$$

$$\therefore y^3(1 - m^3) - 3by^2 + 3b^2y - (b^3 - a^3m^3) = 0.$$

$$\therefore y_1y_2y_3 = \frac{b^3 - a^3m^3}{1 - m^3}.$$

$$\therefore x_1x_2x_3 + y_1y_2y_3 = \frac{a^3 - b^3 + b^3 - a^3m^3}{1 - m^3} = a^3.$$

Let $(\bar{x}, \bar{y})$ be the coordinates of the centroid, and suppose it to lie on the axis of $x$.

Then $\bar{x} = \frac{1}{3}(x_1 + x_2 + x_3)$; $\bar{y} = \frac{1}{3}(y_1 + y_2 + y_3) = 0$,

$$\therefore y_1^3+y_2^3+y_3^3-3y_1y_2y_3=(y_1+y_2+y_3)(y_1^2+y_2^2+y_3^2-y_2y_3-y_3y_1-y_1y_2)$$
$$= 0,$$
$$\therefore y_1 + y_2^3 + y_3^3 = 3y_1y_2y_3 = 3(a^3 - x_1x_2x_3),$$
$$\therefore 3a^3 = x_1^3 + x_2^3 + x_3^3 + y_1^3 + y_2^3 + y_3^3$$
$$= x_1^3 + x_2^3 + x_3^3 + 3a^3 - 3x_1x_2x_3.$$
$$\therefore 0 = x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$$
$$= (x_1 + x_2 + x_3)\{x_1^2 + x_2^2 + x_3^2 - x_2x_3 - x_3x_1 - x_1x_2\}$$
$$= \tfrac{1}{2}(x_1 + x_2 + x_3)\{(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2\}.$$

Now the 2nd factor does not $= 0$, for then we should have $x_1 = x_2 = x_3$,

$$\therefore x_1 + x_2 + x_3 = 0. \quad \therefore \bar{x} = 0, \text{ and } \bar{y} = 0.$$

Similarly if we begin by assuming that $\bar{x} = 0$, we shall deduce $\bar{y} = 0$.
$\therefore$ if the centroid lies on either axis, it is at the origin.

## Paper XCIV.

1. It is easily seen that the given expression is the product of the two factors $(3^{2n+2} - 8n - 9)(3^{2n+3} + 40n - 27)$.

It is proved in XX. No. 2, that $3^{2n+2} - 8n - 9$ is a multiple of $2^6$. By trial we find that $3^{2n+3} + 40n - 2y$ is a multiple of $2^6$ when $n = 1$, and when $n = 2$. Suppose this to be the case when $n = p$. For $n$ write $p + 1$, and subtract the values of the expressions thus obtained. We have $27 \cdot 3^{2p} \cdot 8 + 40 = 8(27 \cdot 9^p + 5)$.

Similarly we see that $27 \cdot 9^p + 5$ is a factor of $9 - 1$, *i.e.* of 8.

$\therefore 3^{2n+3} + 40n - 27$ is a multiple of 64, *i.e.* of $2^6$.

$\therefore$ the given expression is a multiple of $2^{12}$.

5. Let $C$ be the centre of the circle, $Q$, $Q'$, $R$, $R'$ points on the diameter such that $CQ = CQ'$, $CR = CR'$, and let $P$ be any point on the circumference.

Then
$$PQ^2 + PQ'^2 = 2CP^2 + 2CQ^2,$$
$$PR^2 + PR'^2 = 2CP^2 + 2CR^2,$$

$$\therefore a_1^2 - a_2^2 - a^2_{2n-2} + a^2_{2n-1} = 2(CQ^2 - CR^2)$$
$$= 2(CQ + CR)(CQ - CR)$$
$$= 2 \cdot \frac{d}{2n}(n - 1 + n - 2)\frac{d}{2n}$$
$$= \frac{d^2}{2n^2}(2n - 3).$$

First let $n$ be odd. There will then be $\frac{n-1}{2}$ terms of this series, which is an A.P. the common difference being $-4$.

$$\therefore \Sigma = \frac{d^2}{2n^2} \cdot \frac{n-1}{4} \left\{ 2(2n-3) - 4 \cdot \frac{n-3}{2} \right\}$$
$$= \frac{d^2}{2n^2} \cdot \frac{n-1}{4} \cdot 2n = \frac{d^2}{4} \cdot \frac{n-1}{n} = \frac{d^2}{4} \text{ ultimately.}$$

Next let $n$ be even. We shall then have $\frac{n}{2}$ terms.

$$\therefore \Sigma = \frac{d^2}{2n^2} \cdot \frac{n}{4} \left\{ 2(2n-3) - 4 \cdot \frac{n-2}{2} \right\}$$
$$= \frac{d^2}{2n^2} \cdot \frac{n}{4}(2n-2) = \frac{d^2}{4} \cdot \frac{n-1}{4} = \frac{d^2}{4} \text{ ultimately.}$$

Now if we give to $n$ a small odd value, as 7, we see that we have omitted to take in the value of $CP^2$; and if we give $n$ a small even value we see that $CP^2$ has been subtracted twice instead of once. We must $\therefore$ add $CP^2$ to each of the above results, and we obtain

$$\frac{d^2}{4} + \frac{d^2}{4} = \frac{d^2}{2}$$

## Paper XCV.

7. Let $t$ be the time the first particle takes to reach the plane. Let $C$ be the point of projection, and let the vertical through $C$ meet the plane in $D$. Let $CE$ be the direction of projection of the 1st particle. Then the particle will always move in the plane containing $CD$ and $CE$, and $DA$ will be the section of this plane with the given plane.

Then $\frac{1}{2}gt^2 \cos\beta = d$, $\therefore t = \sqrt{\frac{2d \sec\beta}{g}}$.

This is independent of the velocity, $\therefore$ the particles reach the plane simultaneously.

Since $DA$, $DB$ are parallel to $CE$, $CF$, $\therefore$ $ADB = ECF$ = a right angle.

Now $DA$ and $DB$ are the spaces due to the velocities of projection.

$$\therefore AB^2 = AD^2 + DB^2 = (u^2 + v^2)t^2 = \frac{2d}{g}(u^2 + v^2)\sec\beta.$$

## Paper XCVI.

1. $q - r = b - c + \dfrac{(b-c)x^2}{bc}$.

$$\therefore \frac{1}{a}\left\{1 - \frac{q-r}{b-c}\right\} = -\frac{x^2}{abc} = \frac{1}{b}\left\{1 - \frac{r-p}{c-a}\right\} = \frac{1}{c}\left\{1 - \frac{p-q}{a-b}\right\}$$

from symmetry.

$$px = x^2 - yz\,;\ (1)\quad qy = y^2 - zx\,;\ (2)\quad rz = z^2 - xy\,;\ (3).$$

Multiply (1), (2), (3) successively by $y$, $z$, $x$, and then by $z$, $x$, $y$, and add. We thus obtain

$$0 = pxy + qyz + rzx = \frac{q}{x} + \frac{r}{y} + \frac{p}{z} \quad (A).$$

$$0 = pxz + qxy + ryz = \frac{r}{x} + \frac{p}{y} + \frac{q}{z} \quad (B).$$

From $(A)$ and $(B)$ we have

$$\frac{\frac{1}{x}}{p^2 - qr} = \frac{\frac{1}{y}}{q^2 - rp} = \frac{\frac{1}{z}}{r^2 - pq},$$

$\therefore$ substituting in $\dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z} = 0$, we have

$$a(p^2 - qr) + b(q^2 - rp) + c(r^2 - pq) = 0.$$

*2. Let $ABCD$ be the quadrilateral, and let $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = l$. Let $AC$ and $BD$ intersect in $O$. Let $BAC = \lambda$, $CAD = \mu$, $BOC = \phi$.

Then $\Delta$ = area of $ABC = \frac{1}{2}al \sin \lambda$,

$\Delta'$ = area of $ADC = \frac{1}{2}dl \sin \mu$,

$$\therefore \frac{\Delta}{\Delta'} = \frac{a \sin \lambda}{d \sin \mu} = \frac{\sin (\phi + \mu) \sin \lambda}{\sin (\phi - \lambda) \sin \mu} = \frac{\cot \mu + \cot \phi}{\cot \lambda - \cot \phi},$$

$$\therefore (\Delta + \Delta') \cot \phi = \Delta \cot \lambda - \Delta' \cot \mu.$$

But $\Delta \cot \lambda = \frac{1}{2}al \sin \lambda \cot \lambda = \frac{1}{2}al \cos \lambda = \frac{1}{4}(a^2 + l^2 - b^2)$,

$\Delta' \cot \mu = \frac{1}{2}dl \sin \mu \cot \mu = \frac{1}{2}dl \cos \mu = \frac{1}{4}(d^2 + l^2 - c^2)$,

$$\therefore (\Delta + \Delta') \cot \phi = \tfrac{1}{4}(a^2 - b^2 + c^2 - d^2),$$

$$\therefore \text{area } ABCD = \tfrac{1}{4}(a^2 - b^2 + c^2 - d^2) \tan \phi.$$

## Paper XCVII.

1. Let $u = x^3 + 3px^2 + 3qx + r$, and for $u - r$ write $y$.

Then we have to express the condition that the equation

$$x^3 + 3px^2 + 3qx - y = 0$$

has two equal roots. Let the roots be $x_1, x_2, x_3$.

Then
$$\left.\begin{aligned} 2x_1 + x_3 &= -3p; \\ x_1^2 + 2x_1x_3 &= 3q; \\ x_1^2x_3 &= y; \end{aligned}\right\} \quad \therefore x_3 = -(2x_1 + 3p).$$

$$\left.\begin{aligned} \therefore x_1^2 - 2x_1(2x_1 + 3p) &= 3q, \quad \therefore x_1^2 + 2px_1 + q = 0; \\ \text{and} \qquad x_1^2(2x_1 + 3p) &= -y, \quad \therefore 2x_1^3 + 3px_1 + y = 0; \\ &\text{and } 2x_1^3 + 4px_1 + 2qx_1 = 0; \end{aligned}\right\}$$

$\therefore$ by subtraction, $px_1^2 + 2qx_1 - y = 0$

$$\text{and } x_1^2 + 2px_1 + q = 0$$

$$\therefore \frac{x_1^2}{2(py + q^2)} = \frac{-x_1}{pq + y} = \frac{1}{2p^2 - 2q},$$

$$\therefore (pq + y)^2 = 4(p^2 - q)(py + q^2),$$

$$\therefore y^2 - 2y(2p^3 - 3pq) + 4q^3 - 3p^2q^2,$$

$$\therefore y = 2p^3 - 3pq \pm \sqrt{A},$$

where
$$\begin{aligned} A &= (2p^3 - 3pq)^2 - (4q^3 - 3p^2q^2) \\ &= 4p^6 - 12p^4q + 9p^2q^2 - 4q^3 + 3p^2q^2 \\ &= 4(p^6 - 3p^4q + 3p^2q^2 - q^3) \\ &= 4(p^2 - q)^3, \end{aligned}$$

$$\therefore y = 2p^3 - 3pq \pm 2(p^2 - q)^{\frac{3}{2}},$$

$$\therefore u = y + r = 2p^3 - 3pq + r \pm 2(p^2 - q)^{\frac{3}{2}}.$$

NOTE.—The above method also gives a simple solution of the question 'Find the relation between the coefficients in order that a complete cubic equation may have two equal roots.'

4. From any point $P$ on the given circle draw tangents $PQ$, $PR$ to the other circle, centre $S$. Then $SP$ bisects $QR$ at right angles in $Y$, and $SY \cdot SP = SQ^2 = \text{const.}$ $\therefore$ the locus of $Y$ is a circle, and the envelope of $QR$ is a conic which has $S$ for one focus, and the locus of $Y$ for auxiliary circle.

The envelope will be an ellipse or hyperbola according as the circle, centre $S$, is within or without the other circle.

7. Let $P$ be any point on the cycloid, $PT$ the tangent at $P$, $TS$ the corresponding vertical diameter of the generating circle, $PN$ perpendicular to $ST$. Let $R$ denote the pressure on the curve. Then $g \cos \psi$ is the resolved part of the accelerating force of gravity along the normal $PS$. $\therefore \frac{R}{m} - g \cos \psi$ is the total acceleration along the normal.

But $\frac{v^2}{\rho}$ is also the acceleration along the normal. Now $v$ is the velocity due to the height $SN$. $\therefore v^2 = 2g \cdot SN$, and $\rho$, the radius of curvature $= 2SP$,

$$\therefore \frac{v^2}{\rho} = g \cdot \frac{SN}{SP} = g \cos PSN = g \cos TPN = g \cos \psi.$$

$$\therefore \frac{R}{m} - g \cos \psi = \frac{v^2}{\rho} = g \cos \psi. \quad \therefore R = 2mg \cos \psi.$$

Now since the acceleration along the normal is $g \cos \psi$, and along the tangent is $g \sin \psi$, $\therefore$ the resultant acceleration is $g$.

## Paper XCVIII.

6. Let $B$ and $C$ be the feet of the ladders, and suppose the man to go up the ladder $AB$. Let $R$, the reaction at $A$, make an angle $\theta$ with $PQ$ produced, and let $APQ = \phi$.

Resolving horizontally the forces acting on $AB$,

$$T = R\cos\theta. \quad (1)$$

Taking moments about $B$ and $C$,

$$T(l-a)\sin\phi + w\frac{l}{2}\cos\phi + Wb\cos\phi = Rl\sin(\theta+\phi) \quad (2).$$

$$T(l-a)\sin\phi + w\frac{l}{2}\cos\phi = Rl\sin(\phi-\theta) \quad (3).$$

$$\therefore Wb\cos\phi = Rl\{\sin(\theta+\phi) - \sin(\phi-\theta)\},$$

$$= 2Rl\sin\theta\cos\phi.$$

$$\therefore \frac{Wb}{2l} = R\sin\theta \quad (4).$$

$\therefore$ from (1), (3), and (4)

$$T(l-a)\sin\phi + w\frac{l}{2}\cos\phi = Tl\sin\phi - \frac{Wb}{2}\cos\phi,$$

$$\therefore Ta\sin\phi = \frac{Wb+wl}{2}\cos\phi,$$

$$\therefore T = \frac{Wb+wl}{2a}\cot\phi = \frac{Wb+wl}{2a}\cdot\frac{c}{\sqrt{a^2-c^2}}.$$

## Paper XCIX.

4. Let $P$, $P'$ be the orthocentres of $ABC$, $A'BC$, $O$ the centre of the circumcircle, $D$ the middle point of $BC$. Join $A'P$, $AP'$. Then $AP = 2OD = A'P'$, and since $AP$, $A'P'$ are equal and parallel, $AP'$ and $A'P$ bisect each other in $Q$.

Now by XXI. 5, *Ap.* the pedal line of $ABC$ with respect to $A'$ bisects $A'P$. Similarly the pedal line of $A'BC$ with respect to $A$ bisects $AP'$. $\therefore$ these pedal lines intersect in $Q$.

Bisect $OP$ in $N$. Then $N$ is the centre of the nine points' circle of $ABC$, and since $PN = NO$, and $PQ = QA'$, $\therefore NQ = \frac{1}{2}OA' =$ radius of nine points' circle of $ABC$, which $\therefore$ passes through $Q$.

Similarly it may be shewn that $Q$ lies on the circumference of the nine points' circle of $A'BC$.

## Paper C.

2. Multiply (1) by $\sin\theta$, (2) by $\cos\theta$, and subtract.

$$\therefore x \sin a = 2a \cos\beta \cos^2\theta \quad (3).$$

Multiply (1) by $\sin(\theta - a)$, (2) by $\cos(\theta - a)$, and subtract.

$$\begin{aligned}\therefore y \sin a &= 2a\cos(\theta - a)\{\sin(\theta + \gamma)\sin a - \cos\beta\cos\theta\}\\ &= 2a\cos(\theta - a)\{\sin\theta\cos\gamma\sin a + \cos\theta(\sin a\sin\gamma - \cos\beta)\}\\ &= 2a\cos(\theta - a)\{\sin\theta\cos\gamma\sin a + \cos\theta\cos\gamma\cos a\}\\ &= 2a\cos\gamma\cos^2(\theta - a) \quad (4).\end{aligned}$$

From (3), $\cos\theta = \pm\sqrt{\dfrac{x\sin a}{2a\cos\beta}}$, $\therefore \theta = \cos^{-1}\sqrt{P}$.

From (4), $\cos(\theta - a) = \pm\sqrt{\dfrac{y\sin a}{2a\cos\gamma}}$, $\therefore \theta - a = \cos^{-1}\sqrt{Q}$.

$$\therefore a = \cos^{-1}\sqrt{P} - \cos^{-1}\sqrt{Q},$$

$$\therefore \cos a = \sqrt{P}\sqrt{Q} + \sqrt{1-P}\sqrt{1-Q},$$

$$\therefore \cos^2 a - 2\sqrt{PQ}\cos a + PQ = 1 - P - Q + PQ,$$

$$\therefore P + Q - 2\sqrt{PQ}\cos a = \sin^2 a,$$

$$\therefore \frac{x}{\cos\beta} + \frac{y}{\cos\gamma} - 2\sqrt{\frac{xy}{\cos\beta\cos\gamma}}\cdot\cos a = 2a\sin a.$$

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In the solution of V. 7, given on page 14, instead of employing trigonometry we may proceed thus:

Draw $EF$ perpendicular to $AB$, and produce $DB$ to meet the circle, centre $B$ in $G$. Then since $AEB$ is a right angle, $\therefore AE$ touches the circle $DEG$. $\therefore$ the angle $AED = DGE = DEF$.

$$\therefore AD : DF :: AE : EF :: AB : EB.$$

$$\therefore AD \cdot BE = AB \cdot DF, \quad \therefore AD \cdot BD = 2DF \cdot AC. \quad (1)$$

Now $AC \cdot AD = CL^2$, $\therefore AC^2 - AC \cdot AD = AC^2 - CD^2$,

$$\therefore AC \cdot CD = AD \cdot DB.$$

$\therefore$ by (1), $DC = 2DF$, $\therefore DEF = FEC$.

Also $CEB = CBE = 2DGE = 2AED$.

$\therefore BGE = AED = \frac{1}{5}$ of a right angle.

$\therefore ABE = \frac{2}{5}$ of a right angle.

$\therefore ACE = \frac{4}{5}$ of one right angle $= \frac{1}{5}$ of four right angles.

$\therefore AC$ is $\frac{1}{5}$ of the circumference.

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The solution of VI. 6, given on page 16, assumes that the ratio in which $AB$ and $DC$ are divided is the same as the ratio in which $AD$ and $BC$ are divided. When this is not the case, we can prove the theorem as follows.

Since $AE : EB :: DF : FC$. $\therefore$ three parallel planes can be drawn through the lines $AD$, $EF$, $BC$. Similarly three parallel planes can be drawn through $AB$, $GH$, $DC$. Let $FA$, $FB$, $FE$ meet the plane through $GH$ in $K$, $L$, $M$. Join $GK$, $KM$, $MG$; $ML$, $LH$, $HM$. Then $GK$ and $LH$ are parallel to $DF$, and $KM$, $ML$ are parallel to $AB$, also

$$\begin{aligned} GK : DF &:: AK : KF \\ &:: BL : LF \\ &:: HL : CF, \\ \therefore GK : LH &:: DF : CF \\ &:: AE : EB \\ &:: KM : ML, \end{aligned}$$

$\therefore GM$ and $MH$ are in the same straight line. Euc. VI. 32.

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The equation given in LXXI. 1, (3) is incorrectly stated.

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## NOTE I.

### Geometrical Theory of Envelopes.

The propositions which are of the most frequent use in shewing that lines which move subject to certain conditions envelop conic sections are the following, of which 1, 2, 3 are proved in Casey.

1. $S$ is a fixed point, and $P$ any point on a fixed straight line. On $SP$ is taken a point $Q$ such that $SP : SQ$ in a fixed ratio. Then the locus of $Q$ is a straight line.

2. If the point $Q$ be taken so that $SP \,.\, SQ$ = const. the locus of $Q$ is a circle.

3. In 1 and 2, if $P$ moves along the circumference of a fixed circle, the locus of $Q$ is a circle.

4. If $P$ move along a fixed circle, centre $O$, and on a straight line through $S$, making a constant angle with $SP$, a point $Q$ be taken such that $SQ = SP$, the locus of $Q$ is a circle equal to that of $P$.

For draw the diameter through $S$, and at $S$ make the angle $OSO'$ equal to the constant angle $PSQ$, and in the same direction. Make $SO' = SO$. Then $O'$ is a fixed point. Join $OP$, $O'Q$. Then the triangles $OSP$, $O'SQ$ are equal in all respects. $\therefore O'Q = OP$. $\therefore$ the locus of $Q$ is a circle equal to the locus of $P$.

5. In 4, instead of taking $SQ = SP$, if we take $SQ : SP$ in a fixed ratio, or $SP \,.\, SQ$ = const. we see from (3) and (4) that the locus of $Q$ is still a circle.

It will often be found that the question of finding the envelope of a straight line can be put in the following form.

If two straight lines include a constant angle, and one of them always passes through a fixed point, it is required to find the envelope of the other line when the vertex moves along a fixed straight line or circle.

6. Now since the tangent at the vertex of a parabola is the locus of the foot of the perpendicular from the focus on a tangent, $\therefore$ if we have two straight lines at right angles, one of which always passes through a fixed point whilst the vertex moves along a fixed straight line, the other line will envelop a parabola having the fixed point for focus, and the fixed straight line for tangent at the vertex.

7. If in 6 the two lines be inclined at *any* constant angle, the envelope will still be a parabola. Besant, *Parab.* Art. 36.

8. Since the auxiliary circle is the locus of the foot of the perpendicular from the focus of a central conic on a tangent, we see that if we have two straight lines at right angles, one of which always passes through a fixed point whilst the vertex moves along a fixed circle, the other line will envelop a conic having the fixed point for one of its foci, and the fixed circle for its auxiliary circle.

From this it follows that if one of the lines passes through the focus of a given conic, and the angular point moves along a concentric circle, the other line will envelop a confocal conic.

9. Suppose that the lines, instead of being at right angles, include a constant angle. Let $S$ be the fixed point, $PS$, $PQ$ any position of the two lines. Draw $SQ$ perpendicular to $PQ$. Then since the angle $SPQ$ is constant, and $SQP$ is a right angle, $\therefore$ $SP : SQ$ in a constant ratio. But the locus of $P$ is a circle, $\therefore$ by 5, the locus of $Q$ is a circle, and $\therefore$ by 8, the envelope of $PQ$ is a conic having $S$ for a focus and the locus of $Q$ for the auxiliary circle.

Hence we see that if we can find the locus of the vertex, we can at once determine the envelope required. If the locus is a straight line, the corresponding envelope is a parabola. If the locus is a circle the envelope is an ellipse, hyperbola, or point, according as the fixed point is within, without, or on the circumference of the circle which is the locus of the foot of the perpendicular from the fixed point on the line whose envelope is required.

From 7 and 9 we at once deduce the following theorem.

If a triangle of given species has one angular point fixed, and if a second angular point moves along (1) a given line, (2) a given circle, the third will also move along (1) a given line, (2) a given circle; for the side opposite the fixed angle envelopes (1) a parabola, (2) a central conic. See Casey, p. 71; Catalan, *Géom.* p. 81.

## NOTE II.

### Geometrical Maxima and Minima.

It is usual to treat questions of this class as isolated problems, each requiring its own special mode of solution. It will be seen, however, from the examples which are solved in different parts of this work, that problems relating to this subject can in general be treated by the following simple method.

If the variation of a geometrical magnitude be continuous, when the magnitude has its max. or min. value its rate of change is zero; in other words, when a varying quantity is a max. or min. its values in two consecutive positions are equal. $\therefore$ to find when it is a max. or min. we have only to assume two consecutive values to be equal, and we at once deduce the required result.

As illustrations we will work out a few examples.

1. $A$ and $B$ are two points on the same side of a fixed line. It is required to find the point on the line at which $AB$ subtends a max. angle. Let $P$ and $Q$ be two points on the line which are indefinitely near to each other and such that the angle $APB = AQB$. Then a circle will go round $APQB$, and the given line, which passes through the two consecutive points $P$ and $Q$, is a tangent. Hence the construction; through $A$ and $B$ describe a circle touching the given line. The point of contact is the point required.

2. On a given chord as base inscribe the max. triangle in a circle. Assume the inscribed triangle to have the same magnitude for two consecutive points on the curve. The line joining these points is parallel to the base by Euc. i. 37. But the line joining two consecutive points on a curve is a tangent. $\therefore$ the tangent at the vertex is parallel to the base, and the max. triangle is isosceles.

3. Shew that the max. triangle which can be inscribed in a circle is equilateral.

Suppose each of the sides in turn to remain fixed, whilst the opposite vertex moves along the curve. By 2, the triangle is in each case a max. when the tangent at the vertex is parallel to the base. $\therefore$ when all the sides vary, the tangents at the angular points are parallel to the opposite sides, and the triangle is equilateral.

Questions which appear to depend upon the variation of two magnitudes can sometimes be made to depend upon only one.

4. Given two points $A$ and $B$, find the point $P$ on a given line or circle such that $AP^2 + BP^2$ may be a min.

Let $C$ be the middle point of $AB$.

Then $AP^2 + BP^2 = 2AC^2 + 2CP^2$. Thus we have only to find when $CP^2$ is a min. Let $P$ and $P'$ be two consecutive points on the line such that $CP = CP'$. Then since the angle $PCP'$ is very small, $CPP'$ and $CP'P$ are ultimately right angles, and $P$ is the foot of the perpendicular from $C$.

In the case of the circle $CP$ is at right angles to the line joining $P$ and $P'$, *i.e.* to the tangent at $P$. $\therefore$ $P$ is the point of intersection of the circle with the line joining $C$ to the centre.

If $CP$ be produced to meet the circle again in $Q$, $AQ^2 + BQ^2$ is a max.

5. $ABC$ is a semicircle, centre $O$ and diameter $AC$. $ADO$ is another semicircle, centre $O'$, on the opposite side of $AO$. It is required to draw a chord at right angles to $AO$ such that the portion intercepted between the curves shall be a max.

Let $PQ$, $P'Q'$ be two equal consecutive values of the chord. Then $PP'$ and $QQ'$, which are the tangents at $P$ and $Q$, are parallel by Euc. i. 33. $\therefore$ $OP$, $O'Q$ are parallel, and if $PQ$ meets $OO'$ in $N$, the triangles $OPN$, $O'QN$ are similar. $\therefore$ $O'N : ON :: 1 : 2$. $\therefore$ the point $N$ is determined.

6. In LXVII. 3, is given a geometrical proof of the theorem that the max. quadrilateral which can be formed from four given straight lines is inscriptible in a circle. From this it is easy to shew that of all plane rectilineal figures which can be formed from straight lines which are given both in number and length, the max. is inscriptible. For if $A$, $B$, $C$, $D$ be four angular points taken in order, then $ABCD$ is inscriptible. Similarly if $E$ be the next angular point to $D$, the points $BCDE$ will lie on a circle, and these two circles are the same, for they are both described about the triangle $BCD$. By proceeding in this manner we may shew that the theorem holds for any number of straight lines.

If we suppose the number of sides to be increased and their lengths diminished indefinitely, we see that the area of a circle is greater than the area of any other closed figure having the same perimeter.

7. Again, since $(ax + by)^2 + (bx - ay)^2 = (a^2 + b^2)(x^2 + y^2)$,

$\therefore$ if $ax + by$ be given, and $a$ and $b$ be constants, $x^2 + y^2$ is a min. when $\frac{x}{a} = \frac{y}{b}$. The geometrical meaning of this gives a solution of the question

'In the base of a triangle find a point such that the sum of the squares on the perpendiculars drawn from it to the sides shall be a min.'

If the point be within the triangle, $x, y, z$ its perpendicular distances from the sides, we at once deduce that $x^2 + y^2 + z^2$ is a min. when $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, and the value of the min. sum is evidently

$$\frac{4\Delta^2}{a^2 + b^2 + c^2}.$$

It may be noticed that the point thus determined is the 'Symmedian' point. See Paper LXVIII., 4.

8. By considering

$$(ax + by)^2 - (ax - by)^2 = 4abxy,$$

we see that the middle point of the base of a triangle is such that the product of the perpendiculars from it to the sides is a min.; and we deduce that the point within a triangle such that the product of the perpendicular distances from it to the sides is a min. is the centroid of the triangle, and that the value of the min. product is $\frac{\Delta^3}{abc}$.

For further treatment of this subject, see an elementary treatise by the present author on Geometrical and Algebraical Max. and Min. without the aid of Differential Calculus, where it is shewn that the problem of finding the max. or min. values of geometrical magnitudes depending only on the point line and circle is reduced to the much simpler one of finding its positions of symmetry.

THE END.

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